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PhD Preliminary Examination in Analysis

Department of Mathematics

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1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers given by

$$x_1 > \sqrt{2} \quad \text{and} \quad x_{n+1} = \frac{2+x_n}{1+x_n}, \quad n = 1, 2, \ldots$$

Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and find its limit.

2. Let $f : [0,1] \to \mathbb{R}$ and $\alpha : [0,1] \to \mathbb{R}$ be two real valued functions on the interval $[0,1]$. Suppose that the function f is monotone and the function α is increasing and continuous. Prove that f is integrable with respect to α on $[0,1]$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 0$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ defined for any $ty \in \mathbb{R}$ by

$$f_n(t) = n \int_0^t \left[f\left(s + \frac{1}{n}\right) - f(s) \right] ds.$$

Prove that the sequence $\{f_n\}$ converges to f on the closed interval $[0,1]$. Does f_n uniformly converge to f on $[0,1]$? Justify your answer.

4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ which converges to a function $f : \mathbb{R} \to \mathbb{R}$, that is, for any $x \in \mathbb{R}$, $f(x) = \lim_{n\to\infty} f_n(x)$. Suppose that for any $x \in \mathbb{R}$,

$$f_1(x) \le f_2(x) \le f_3(x) \le \cdots \le f_n(x) \le f_{n+1}(x) \le \cdots.$$

Prove that for any real number $x \in \mathbb{R}$ and any sequence $\{x_k\}_{k=1}^{\infty}$ converging to x (that is, $x = \lim_{k\to\infty} x_k$), the following holds:

$$\liminf_{k\to\infty} f(x_k) \ge f(x).$$

5. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that for all $z \in \mathbb{C}$

$$|f'(z)| \le |z|.$$

Show that $f(z) = a + bz^2$ with $|b| \le \frac{1}{2}$.

6. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that $\operatorname{Im} f(z) \le 0$ for any $z \in \mathbb{C}$. Prove that f is constant.

7. Let P be a polynomial of degree n. Let C be a sufficiently large circle centered at the origin and oriented counterclockwise. Show that

$$I = \oint_C \frac{dz}{2\pi i} \frac{P'(z)}{P(z)} = n$$

8. Let $\alpha, \beta \in \mathbb{R}$ be two real numbers. Use principal value integrals to show that

$$\int_0^\infty \frac{dx}{x^2} \left[\cos(\alpha x) - \cos(\beta x)\right] = -\frac{\pi}{2} \left(|\alpha| - |\beta|\right) .$$