Ph.D. Preliminary Examination in Numerical Analysis
Department of Mathematics
New Mexico Institute of Mining and Technology
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1. This exam is four hours long.
2. Work out all six problems.
3. Start the solution of each problem on a new page.
4. Number all of your pages.
5. Sign your name on the following line and put the total number of pages.
6. Use this sheet as a coversheet for your papers.

NAME: ________________________________

No. of pages:___________

Problem 1.

a) Develop the Taylor's method of order 3 for the following initial value problem:

$$x' = x^2 + t, \qquad x(t_0) = x_0.$$

b) Given X_i, the numerical solution at $t = t_i$, show how to compute the solution X_{i+1} at $t_{i+1} = t_i + h$.

Problem 2.

Suppose that f is a smooth function of a single variable. Using values $f(x)$, $f(x+h)$, and $f(x+3h)$ and the Taylor series formula, derive the best finite difference approximation for $f''(x)$, and determine the order of accuracy of the approximation.

Problem 3.

Determine the degree of exactness of the following quadrature formula:

$$\int_{-1}^{1} f(x)dx \approx Q(f) \equiv \frac{7}{15}f(-1) + \frac{16}{15}f(0) + \frac{7}{15}f(1) + \frac{1}{15}f'(-1) - \frac{1}{15}f'(1).$$

In fact, the quadrature formula can be obtained by integrating the Hermite interpolant $H(x)$ of function $f(x)$ at points $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$; that is,

$$Q(f) = \int_{-1}^{1} H(x)\, dx.$$

Using the following error formula of the corresponding Hermite interpolant

$$f(x) = H(x) + \frac{\omega(x)}{6!} f^{(6)}(\xi(x)),$$

where

$$\omega(x) = x^2(x^2-1)^2,$$

and the weighted mean value theorem for integrals:

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx, \ \xi \in (a,b), \ \text{sign}(g) = \text{const.},$$

obtain the error formula of the quadrature

$$\int_{-1}^{1} f(x)df = \int_{-1}^{1} H(x)dx + \frac{f^{(6)}(\bar{\xi})}{4725}, \ \bar{\xi} \in (0,1).$$

Problem 4. Describe the Newton method and the Secant method for solving the scalar equation $f(x) = 0$ for $x \in [a,b]$. Discuss advantages and disadvantages of the methods.

Problem 5. Consider an m by n real matrix A, and let B be the matrix of the same size with entries $b_{i,j} = |a_{i,j}|$. Let $\| \cdot \|_2$ be the matrix 2-norm. Show that

$$\|A\|_2 \leq \|B\|_2. \tag{1}$$

Hint: prove the bound $\|Ax\|_2 \leq \|Bx\|_2$.

Problem 6. Let

$$M = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

be a positive definite matrix with square diagonal blocks. The matrix

$$N = C - B^t A^{-1} B$$

is known as the Schur complement of block A in M. Prove that the Schur complement N is positive definite.

Hint: Consider $x^t M x$, and let vector

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

be partitioned in accordance with the partitioning of matrix M. For any vector x_2, let

$$x_1 = D x_2$$

for some matrix D. Find a scalar c such that, for $D = cA^{-1}B$, it follows that

$$x^t M x = x_2^t N x_2.$$