Numerical Analysis Qualifying Exam
Mathematics Department, New Mexico Tech
Fall 2011

(Answer all six problems)

1. Suppose that the sequence of points, x_k, generated by Newton's method converges to a root $f(x^*) = 0$. You may also assume f is twice continuously differentiable and that $f'(x^*) \neq 0$. Let $e_k = x_k - x^*$. Show that
$$\lim_{k\to\infty} \left|\frac{e_{k+1}}{e_k^2}\right| = \left|\frac{f''(x^*)}{2f'(x^*)}\right|.$$
This shows that the method has quadratic convergence to the root.

2. Suppose that you have been given the LU factorization of an n by n matrix A,
$$PA = LU$$
where P is an n by n permutation matrix, L is an n by n lower triangular matrix and U is an n by n upper triangular matrix. You may assume that A is nonsingular.

   Describe how you would use the LU factorization of A to solve a system of equations $Ax = b$. Your solution should involve solving one lower triangular system of equations involving L and one upper triangular system of equations using U. Show that the computed solution satisfies the original equation.

3. Let $A \in R^{m\times n}$, with $m \geq n$, be given by $a_{i,j} = t_i^{j-1}$ where $\{t_i\}_{i=1}^m$ are distinct real numbers. Use the fundamental theorem of algebra to prove that A is full rank.

   Hint: We know that the $(n-1)^{th}$ degree polynomial $p(t) = \sum_{j=1}^n c_j t^{j-1}$ has exactly $n-1$ roots. Assume the columns of A are linearly dependent and obtain a contradiction.

4. Let Q be an n by n orthogonal matrix, i.e. the columns of Q are orthonormal.

   (a) Show that $Q^TQ = QQ^T$.

   (b) Show that for any x and y in R^n, the standard inner product is preserved, i.e. $\langle Qx, Qy\rangle = \langle x, y\rangle$, and, in particular, the Euclidean norm is preserved, i.e. $\|Qx\|_2 = \|x\|_2$.

   (c) Given $A \in R^{m\times n}$ of rank $n \leq m$, use the fact that a $m\times m$ orthogonal matrix Q can be found so that $Q^TA = R = \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix}$ where $\tilde{R}$ is a nonsingular $n\times n$ upper triangular matrix to solve the least squares problem: minimize $\|b - Ax\|_2$ over all $x \in R^n$.

5.

(a) Derive the 2 point Gaussian quadrature formula.

(b) Approximate the following integral using the two-point Gaussian quadrature rule and find the relative error of the approximation:

$$\int_0^2 \sin(x)\ dx$$

6. Determine the parameters a, b, c, d, and e so that the following function $S(x)$ is a natural cubic spline:

$$S(x) = \begin{cases} a + b(x-1) + c(x-1)^2 + d(x-1)^3, & x \in [0,1], \\ (x-1)^3 + ex^2 - 1, & x \in [1,2], \end{cases}$$