DE Preliminary Exam Spring 2019

Do all the problems on your own and show all your work for full credit.

ODES

1. Consider the system (Application of Poincare-Bendixson Theorem)

$$\begin{aligned}\dot{x} =& -y + x - x^3,\\ \dot{y} =& x + y + y^3.\end{aligned}$$

   (i) Show that the system has a periodic orbit in the annular region

$$A = \{\mathbf{x} \in \mathbf{R}^2 | 1 < |\mathbf{x}| < \sqrt{2}\}.$$

   (ii) Show that there is at least one stable limit cycle in A.

2. Consider the system

$$\begin{aligned}x' =& \varepsilon x - y - xz,\\ y' =& x + \varepsilon y - yz,\\ z' =& -z + (1+\varepsilon)(2\varepsilon + 1)(x^2 + y^2)\end{aligned}$$

   where ε is a real number close to zero.

   (i) Classify the equilibrium $(0, 0, 0)$ for the linearized system.

   (ii) For $\varepsilon \geq 0$ small, approximate a center manifold $W^c_\varepsilon(0)$ as a graph of a function of the form

$$z = ax^2 + bxy + cy^2 + \cdots$$

   (Note that a, b and c may depend on ε.)

3. Analyze

$$x_{n+1} = \frac{1}{x_n} + \frac{x_n}{2} - 1.$$

PDES

1. Determine the solution of the initial value problem (unidirectional, nonlinear wave equation):

$$u_t + uu_x = 0, \quad u(x,0) = \begin{cases} 2+x, & x \in [-2,0), \\ 2-x, & x \in [0,1), \\ 0, & \text{otherwise.} \end{cases}$$

In particular, determine when the wave breaks **and** the equation of the shock(s). Summarize your final solution in the form of an $x-t$ plane graph, indicating: $u = u_1(x,t)$ in region$_1$, $u = u_2(x,t)$ in the region$_2$, etc.

2. (a) Use the finite cosine transform to find the solution of the BVP

$$u_{xx} + u_{yy} = h(x), \; 0 < x < 1, \; 0 < y < b$$

with the boundary conditions

$$u_x(0,y) = 0, \; u_x(1,y) = 0, \; \text{and} \; u(x,0) = 0, \; u(x,b) = 0.$$

(b) Find the (simplified) solution for the case when $h(x) = \cos(\pi x)$.

Note that for part (a), the solution will involve an infinite sum. For part (b), you should be able to find a *closed form* solution (**not** a ful, infinite series.)

3. Solve the BVP

$$u_{tt} = c^2 u_{xx} + g(x), \qquad x > 0, \; t \geq 0$$

with initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0$$

and boundary condition

$$u(0,t) = 0.$$