NEW MEXICO INSTITUTE OF MINING AND TECHNOLOGY

DRAWDOWN DISTRIBUTION AROUND A WELL PARTIALLY PENETRATING A THICK LEAKY AQUIFER

by

JOHN C. HALEPASKA

Submitted to the faculty of the
New Mexico Institute of Mining & Technology
in partial fulfillment of the requirements
for the degree of Master of Science
in Ground Water Hydrology

May 1966

Table of Contents

ABSTRACT

Steady-state and unsteady-state solutions are found for the potential distribution around wells that partially penetrate thick leaky artesian aquifers. Several graphs illustrating the tabulated function are contained in the thesis. The unsteady-state solutions depend on a function that has to be tabulated if it is to be useful in practical application. The steady-state solutions are obtained in terms of a function that is herein tabulated for a practical range of the parameters. Approximate relations for this function that are useful in practical computations are presented also.

Acknowledgement

The writer is grateful to Dr. Mahdi S. Hantush, Dr. Gerardo Wolfgang Gross and Professor C. E. Jacob for their invaluable help while serving as thesis advisors. Thanks are also extended to Mr. Merle Hanson who programmed part of the tabulated material, to Col. Pennington of Kirtland Air Force Base who donated the computer time, and to Susie Nevergold who typed the manuscript.

Tables

Table 1. $H(\beta) = \frac{\pi}{2} [H_0(\beta) - Y_0(\beta)]$

Table 2. $\frac{s}{Q/4\pi K l'}$ versus β for $l'/a = 1$

Figures

INTRODUCTION

General

Drawdown equations are available for both confined and unconfined aquifers and for completely and partially penetrating wells. The leaky aquifer constitutes a transition case between the perfectly confined aquifer and the unconfined aquifer. Leaky-aquifer theory has been developed for aquifers of finite thickness and completely and partially penetrating wells.

A common occurrence is a thick leaky aquifer partially penetrated by wells. No analytical description of thick leaky aquifers partially penetrated by a well has been given. An aquifer is "thick" when the effect of pumping through a partially penetrating well is negligible at the aquifer's bottom.

Previous Work

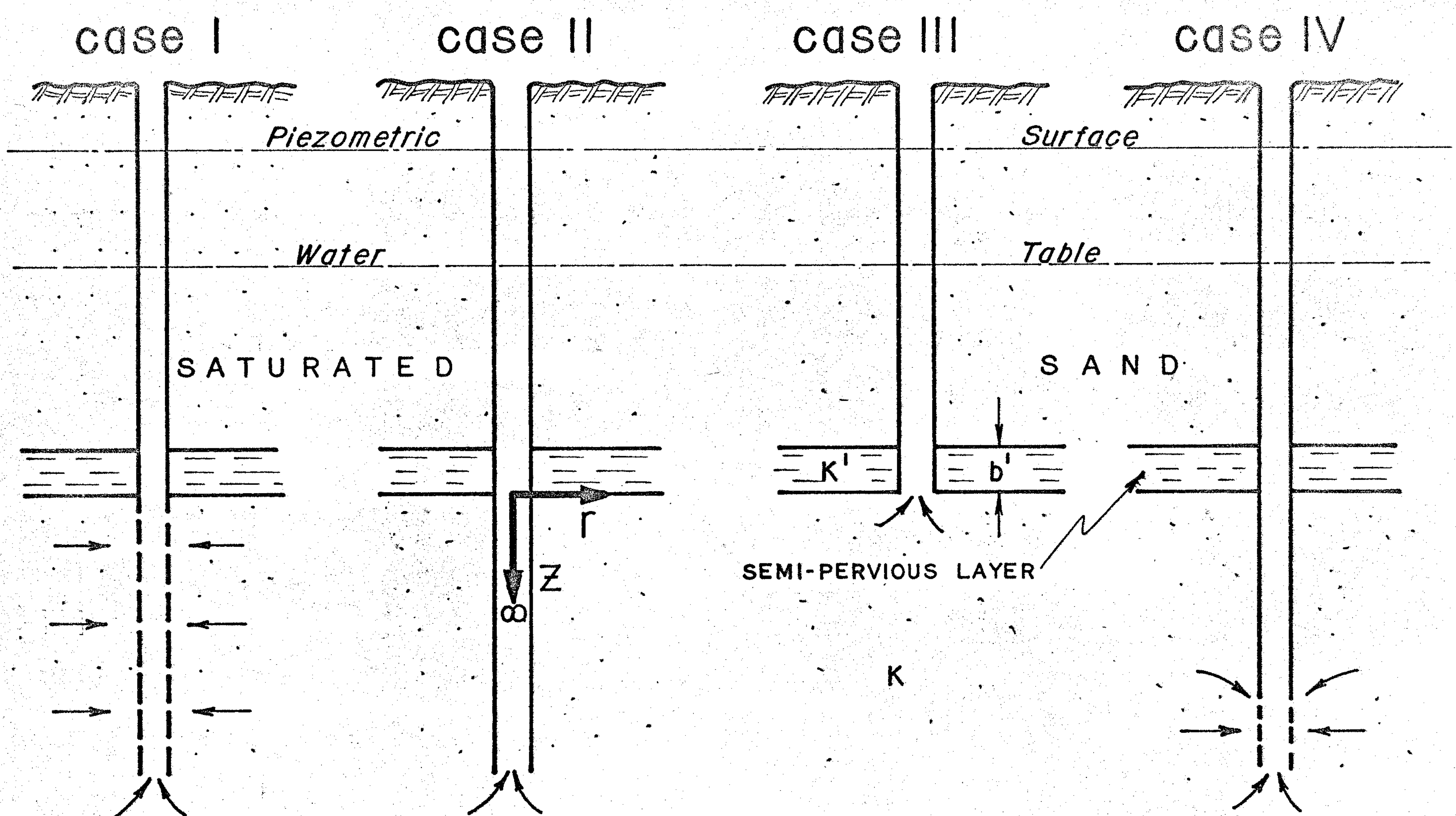
Leaky aquifers have been recognized since before the turn of the century (Jacob, 1946). It was only in the thirties, however, that theoretical solutions for various leaky cases began to appear.

The first readily usable steady-state solution for a well completely penetrating a leaky aquifer was introduced by Jacob in 1946.

Since that time leaky-aquifer theory has greatly increased in sophistication (Hantush & Jacob, 1954, and 1955; Hantush, 1956, 1957, 1959, and 1960). In 1955, Hantush and Jacob developed the theory of non-steady radial flow in an infinite leaky aquifer. This series of solutions, however, was for completely penetrating wells. In 1957, Hantush developed the theory of non-steady flow to a well partially penetrating an infinite leaky aquifer. In this theoretical development the leaky boundary is incorporated in the differential equation. The leakage is then considered to be generated within the aquifer at a rate directly proportional to the drawdown at any place. The error involved in the use of this equation increases as the ratio of penetration to total thickness decreases.

Purpose

The purpose of this study is to develop a solution for the case of a well partially penetrating a thick leaky confined aquifer, expressing the leakage across the interface as a boundary condition.



DIAGRAMATIC REPRESENTATION OF CASES TREATED

Fig. I

Flow Systems

The following flow systems are considered:

1) Wholly perforated wells of penetration l'. This case is treated as purely axisymmetric flow of water discharging into a cylinder of length l'.

2) Wells of penetration l' and zero perforation. This case is considered as the purely radial flow of water into a spherical cavity of the same radius as the well.

3) Wells of zero penetration. This case is treated as the purely radial flow of water to a hemispherical cavity, the same radius as the well.

4) Wells of penetration l' and perforation $(l'-d)$. This case is treated by superposing a recharging well of penetration d on a discharging well of penetration l', both wholly perforated.

Theory

General Assumptions and Boundary Conditions

In an aquifer which is partially penetrated by a well, if the effects of pumpage are negligible at the lower confining layer of the aquifer, it is considered an effectively infinitely thick aquifer. A leaky aquifer is treated as one receiving "linear" leakage, or leakage that is directly proportional to the drawdown, at the interface between the aquifer and the overlying leaky semipervious layer.

The system under consideration is that of a "thick" aquifer overlain by a thin semipervious leaky layer, which is in turn overlain by a pervious saturated bed. The assumptions made are: (1) the head above the semipervious leaky layer is constant; (2) storage in the semipervious layer is negligible; (3) leakage through the semipervious layer is vertical and proportional to the drawdown of the piezometric surface at the leaky interface--such a condition is realized if the hydraulic conductivity of the semipervious layer is very much less than that of the thick layer; (4) the well is discharging at a constant rate; (5) flow to the well is axisymmetric; (6) the aquifer is homogeneous, isotropic, infinite in areal extent, and "thick"; (7) initially the drawdown distribution is zero throughout the aquifer; (8) the hydraulic conductivity and specific storage of the aquifer remain constant in space and time.

Differential Equation of Motion

The general differential equation* of motion is

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}$$

In cylindrical coordinates, the equation is

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}$$

For axisymmetric flow, the equation reduces to

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}$$

Methods of Analysis

Several procedures can be employed in solving the problems treated below. The one presented is considered to be the simplest. The methods of Laplace, Fourier, and Hankel transformations are used to find the drawdown distribution around a fully perforated cylinder draining a thick partially penetrated aquifer. The transformations and integral relations used in the treatment are given in Appendices I and II.

---

* See 'Symbols,' p. 44.

Case 1. Flow to a well, penetration l', perforation l', constant discharge.

(a) Unsteady state

The flow system being radial, with the origin taken at the intersection of the axis of the well and the bottom of the semipervious layer, the boundary value problem in (r, z, t) is (see Fig. 1)

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{\nu}\frac{\partial s}{\partial t} \qquad \text{--- (1)}$$

$$\frac{\partial s(r,0,t)}{\partial z} = \frac{s(r,0,t)}{a} \qquad \text{--- (2)}$$

$$\frac{\partial s(r,\infty,t)}{\partial z} = 0 \qquad \text{--- (3)}$$

$$s(r,z,0) = 0 \qquad \text{--- (4)}$$

$$s(\infty,z,t) = 0 \qquad \text{--- (5)}$$

$$\lim_{r\to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2\pi K l'} \qquad 0 < z < l' \qquad \text{--- (6)}$$

$$= 0 \qquad z > l'$$

Using the Laplace transform with respect to t and applying equation (4), the transformed boundary value problem in (r,z,p) is

$$\frac{\partial^2 \bar{s}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}}{\partial r} + \frac{\partial^2 \bar{s}}{\partial z^2} = \frac{p}{\nu}\bar{s} \qquad \text{--- (7)}$$

$$\frac{\partial \bar{s}(r,0,p)}{\partial z} = \frac{\bar{s}(r,0,p)}{a} \qquad - - - (8)$$

$$\frac{\partial \bar{s}(r,\infty,p)}{\partial z} = 0 \qquad - - - (9)$$

$$\bar{s}(\infty,z,p) = 0 \qquad - - - (10)$$

$$\lim_{r\to 0} r \frac{\partial \bar{s}}{\partial r} = -\frac{Q}{2\pi K l' p} \qquad 0 < z < l' \qquad - - - (11)$$

$$= 0 \qquad z > l'$$

Using the Fourier cosine transform with respect to z and applying boundary conditions (8) and (9), the transformed boundary value problem in (r,ω,p) is

$$\frac{\partial^2 \bar{s}_c}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{s}_c}{\partial r} - \omega^2 s_c - \frac{2}{\pi}\frac{\partial s(r,0,p)}{\partial z} = \frac{p\bar{s}_c}{\nu} \qquad - - - (12)$$

$$\bar{s}_c(\infty,\omega,p) = 0 \qquad - - - (13)$$

$$\lim_{r\to 0} r \frac{\partial s_c}{\partial r} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\omega}\,[\sin\omega l']\left[-\frac{Q}{2\pi K l' p}\right] \qquad - - - (14)$$

Using the Hankel transform of zero order with respect to r and using boundary conditions (13) and (14), the transformed boundary value problem in $(\bar{\alpha},\omega,p)$ is

$$\frac{A}{p}\sqrt{\frac{2}{\pi}}\frac{1}{\omega}\,[\sin \omega\, l'] - (p/\nu + \omega^2 + \bar{\alpha}^2)\ \bar{V}_c = \sqrt{\frac{2}{\pi}}\frac{1}{a}\,\bar{V}(\bar{\alpha},0,p) \qquad (15)$$

where

$$A = \frac{Q}{2\pi K l'}$$

Solving for $\bar{V}_c$

$$\bar{V}_c = \frac{A}{p}\sqrt{\frac{2}{\pi}}\frac{\text{Sin }\omega\, l'}{\omega(p/\nu+\omega^2+\bar{\alpha}^2)} - \frac{1}{a}\sqrt{\frac{2}{\pi}}\frac{\bar{V}(r,0,p)}{p/\nu+\omega^2+\bar{\alpha}^2} \qquad (16)$$

Using the Fourier cosine inversion formula, equation (16) becomes

$$\bar{V} = \frac{A}{p}\frac{2}{\pi}\int_0^\infty \frac{\text{Sin}(\omega\, l')\cos(\omega z)\,d\omega}{\omega(p/\nu+\omega^2+\bar{\alpha}^2)} - \frac{1}{a}\frac{2}{\pi}\,\bar{V}(\bar{\alpha},0,p)\int_0^\infty \frac{\text{Cos }\omega z\ d\omega}{p/\nu+\omega^2+\bar{\alpha}^2} \qquad (17)$$

Rewriting equation (17)

$$\bar{V} = \frac{A}{p\pi}\int_0^\infty \frac{[\text{Sin }\omega(l'+z) + \text{Sin }\omega(l'-z)]\,d\omega}{\omega(p/\nu+\omega^2+\bar{\alpha}^2)} - \frac{2\bar{V}(\bar{\alpha},0,p)}{a\pi}\int_0^\infty \frac{\text{Cos }\omega z\ d\omega}{p/\nu+\omega^2+\bar{\alpha}^2} \qquad (18)$$

Using the inverse Fourier sine and cosine transforms from Appendix I, equation (18) reduces to (if $l' > z$)

$$\bar{V} = \frac{A}{2p}\,\frac{\left(2-e^{-(l'+z)\sqrt{p/\nu+\bar{\alpha}^2}}-e^{-(l'-z)\sqrt{p/\nu+\bar{\alpha}^2}}\right)}{(p/\nu + \bar{\alpha}^2)} - \frac{\bar{V}(\bar{\alpha},0,p)}{a}\,\frac{e^{-z\sqrt{p/\nu+\bar{\alpha}^2}}}{\sqrt{p/\nu+\bar{\alpha}^2}} \qquad (19)$$

Evaluating $\bar{V}(\bar{\alpha},0,p)$ of equation (19)

$$\bar{V}(\bar{\alpha},0,p) = \frac{A}{p}\,\frac{\left(1-e^{-l'\sqrt{p/\nu+\bar{\alpha}^2}}\right)}{\sqrt{p/\nu+\bar{\alpha}^2}\left(\frac{1}{a}+\sqrt{p/\nu+\bar{\alpha}^2}\right)} \tag{20}$$

Substituting equation (20) into equation (19) yields (if $l' > z$)

$$\bar{V} = \frac{A}{2p}\,\frac{\left(2-e^{-(l'+z)\sqrt{p/\nu+\bar{\alpha}^2}}-e^{-(l'-z)\sqrt{p/\nu+\bar{\alpha}^2}}\right)}{(p/\nu+\bar{\alpha}^2)} - \frac{A}{p}\,\frac{\frac{1}{a}\left(e^{-z\sqrt{p/\nu+\bar{\alpha}^2}}-e^{-(l'+z)\sqrt{p/\nu+\bar{\alpha}^2}}\right)}{(p/\nu+\bar{\alpha}^2)\left(\frac{1}{a}+\sqrt{p/\nu+\bar{\alpha}^2}\right)} \tag{21}$$

Using the inverse Hankel transform from Appendix I, equation (21) becomes

$$\bar{s} = \frac{A}{2}\int_0^\infty \frac{\left(2-e^{-(l'+z)\sqrt{p/\nu+\bar{\alpha}^2}}-e^{-(l'-z)\sqrt{p/\nu+\bar{\alpha}^2}}\right)\bar{\alpha}J_0(\bar{\alpha}r)\,d\bar{\alpha}}{p(p/\nu+\bar{\alpha}^2)} - A\int_0^\infty \frac{\frac{1}{a}\left(e^{-z\sqrt{p/\nu+\bar{\alpha}^2}}-e^{-(l'+z)\sqrt{p/\nu+\bar{\alpha}^2}}\right)\bar{\alpha}J_0(\bar{\alpha}r)\,d\bar{\alpha}}{p(p/\nu+\bar{\alpha}^2)\left(\frac{1}{a}+\sqrt{p/\nu+\bar{\alpha}^2}\right)} \tag{22}$$

Rearranging equation (22) gives

$$\bar{s} = \frac{A\nu}{2}\int_0^\infty \frac{\left(2-e^{-\frac{(l'+z)}{\sqrt{\nu}}\sqrt{p+\nu\bar{\alpha}^2}}-e^{-\frac{(l'-z)}{\sqrt{\nu}}\sqrt{p+\nu\bar{\alpha}^2}}\right)\bar{\alpha}J_0(\bar{\alpha}r)\,d\bar{\alpha}}{p(p+\nu\bar{\alpha}^2)} - A\nu\int_0^\infty \frac{\frac{\sqrt{\nu}}{a}\left(e^{-\frac{z}{\sqrt{\nu}}\sqrt{p+\nu\bar{\alpha}^2}}-e^{-\frac{(l'+z)}{\sqrt{\nu}}\sqrt{p+\nu\bar{\alpha}^2}}\right)\bar{\alpha}J_0(\bar{\alpha}r)\,d\bar{\alpha}}{p(p+\nu\bar{\alpha}^2)\left(\frac{\sqrt{\nu}}{a}+\sqrt{p+\nu\bar{\alpha}^2}\right)} \tag{23}$$

Using the convolution and translation theorems from Appendix I on the integrand of (23) term by term yields

$$\bar{s} = \frac{A\nu}{2}\int_0^\infty \left\{ 1 * \left(2 - L^{-1}\left[\frac{e^{-(l'+z)E}}{p} \right] - L^{-1}\left[\frac{e^{-(l'-z)E}}{p} \right] - 2L^{-1}\left[\frac{\frac{\sqrt{\nu}}{a}\, e^{-zE}}{p\left(\frac{\sqrt{\nu}}{a} + \sqrt{p}\right)} \right] + 2L^{-1}\left[\frac{\frac{\sqrt{\nu}}{a}\, e^{-(l'+z)E}}{\left(\frac{\sqrt{\nu}}{a} + \sqrt{p}\right)} \right] \right) e^{-\nu\bar{\alpha}^2 t} \right\} \bar{\alpha} J_0(\bar{\alpha} r)\, d\bar{\alpha} \qquad (24)$$

Where $E = \frac{\sqrt{p}}{\sqrt{\nu}}$

Using Appendix II, equations (3, 4, and 6), equation (24) reduces to

$$s = \frac{A}{2}\int_0^t F \left\{ 1 - \frac{1}{2}\,\mathrm{erfc}\left(\frac{l'+z}{\sqrt{4\nu\tau}}\right) - \frac{1}{2}\,\mathrm{erfc}\left(\frac{l'-z}{\sqrt{4\nu\tau}}\right) - \left[- e^{\frac{z}{a}}\, e^{\frac{\nu\tau}{a^2}}\,\mathrm{erfc}\left(\frac{\sqrt{\nu\tau}}{a} + \frac{z}{\sqrt{4\nu\tau}}\right) + \mathrm{erfc}\left(\frac{z}{\sqrt{4\nu\tau}}\right) \right] + \left[-e^{\frac{l'+z}{a}}\, e^{\frac{\nu\tau}{a^2}}\,\mathrm{erfc}\left(\frac{\sqrt{\nu\tau}}{a} + \frac{l'+z}{\sqrt{4\nu\tau}}\right) + \mathrm{erfc}\left(\frac{l'+z}{\sqrt{4\nu\tau}}\right) \right] \right\} d\tau \qquad (25)$$

Where $F = \frac{e^{-\frac{r^2}{4\nu\tau}}}{\tau}$ and $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ in which erf and erfc(x) are the error function and complementary error function respectively.

Simplifying and making the substitution $y = \frac{r^2}{4\nu\tau}$ and $\frac{d\tau}{\tau} = -\frac{dy}{y}$, equation (25) reduces to

$$s = \frac{Q}{4\pi K l'} \left[2 \int_u^\infty \frac{e^{-y}}{y} e^{\frac{z}{a}} e^{\frac{r^2}{4a^2y}} \operatorname{erfc}\left(\frac{r}{2a\sqrt{y}} + \frac{z\sqrt{y}}{r} \right) dy \right.$$

$$- 2 \int_u^\infty \frac{e^{-y}}{y} e^{\frac{l'+z}{a}} e^{\frac{r^2}{4a^2y}} \operatorname{erfc}\left(\frac{r}{2a\sqrt{y}} + \frac{(l'+z)\sqrt{y}}{r} \right) dy \qquad (26)$$

$$\left. + 2 M(u,\frac{z}{r}) + M(u,\frac{l'-z}{r}) - M(u,\frac{l'+z}{r}) \right]$$

Where $u = \frac{r^2}{4\nu\tau}$ and $M(u,c)$ is a function available in tabular form (Hantush, 1961) and is defined by

$$M(u,c) = \int_u^\infty \frac{e^{-y}}{y} \operatorname{erf}(c\sqrt{y})\, dy$$

where c is an argument in terms of l', z and r.

Equation (26) is the unsteady-state final solution when $l' > z$. When $l' < z$, a process similar to that followed previously will, by substituting (20) in (18) with $l' < z$, yield the same result, namely equation (26).

(b) Steady-state

Using the final value theorem from Laplace transforms (Carslaw and Jaeger, 1963) which is stated $\lim_{t\to\infty} f(t) = \lim_{p\to 0} pf(p)$, if $pf(p)$ is analytic on the axis of imaginaries and in the right half-plane, equation (22) reduces to the form

$$s = \frac{A}{2} \int_0^\infty \frac{(2 - e^{-(l'+z)\alpha} - e^{-(l'-z)\alpha})\, J_0(\bar{\alpha} r)\, d\bar{\alpha}}{\bar{\alpha}}$$

$$- A \int_0^\infty \frac{\frac{1}{a}\,(e^{-z\bar{\alpha}} - e^{-(l'+z)\bar{\alpha}})\, J_0(\bar{\alpha} r)\, d\bar{\alpha}}{\bar{\alpha}(\frac{1}{a} + \bar{\alpha})} \qquad (27)$$

Using partial fractions, equation (27) becomes

$$s = \frac{A}{2} \Big[\int_0^\infty \frac{(1-e^{-(l'+z)\bar{\alpha}})\ J_0(\bar{\alpha}r)\,d\bar{\alpha}}{\bar{\alpha}} + \int_0^\infty \frac{(1-e^{-(l'-z)\bar{\alpha}})}{\bar{\alpha}} J_0(\bar{\alpha}r)\,d\bar{\alpha}$$

$$+ 2\int_0^\infty \frac{(1-e^{-z\bar{\alpha}})\ J_0(\bar{\alpha}r)\,d\bar{\alpha}}{\bar{\alpha}} - 2\int_0^\infty \frac{(1-e^{-(l'+z)\bar{\alpha}})}{\bar{\alpha}} J_0(\bar{\alpha}r)\,d\bar{\alpha} \tag{28}$$

$$+ 2\int_0^\infty \frac{(e^{-z\bar{\alpha}}-e^{-(l'+z)\bar{\alpha}})\ J_0(\bar{\alpha}r)\,d\bar{\alpha}}{\frac{1}{a}+\bar{\alpha}} \Big]$$

Using Appendix II (equation 7) and manipulating, equation (28) reduces to

$$s = \frac{A}{2} \Big[c + 2\int_0^\infty \frac{(e^{-z\bar{\alpha}}-e^{-(l'+z)\bar{\alpha}})\ J_0(\bar{\alpha}r)\,d\bar{\alpha}}{\frac{1}{a}+\bar{\alpha}} \tag{29}$$

where $c = [\sinh^{-1}\left(\frac{l'-z}{r}\right) - \sinh^{-1}\left(\frac{l'+z}{r}\right) + 2\sinh^{-1}\left(\frac{z}{r}\right)]$

Manipulating further, equation (29) reduces to

$$s = \frac{Q}{4\pi K l'} [\ c + 2\ I\ (\alpha,\ \beta) - 2\ I\ (\gamma,\ \beta)\ ] \tag{30}$$

Where

$$I(\alpha,\beta) = e^{\alpha}\int_\alpha^\infty \frac{e^{-\nu}d\nu}{\sqrt{\beta^2+\nu^2}}$$

$$I(\gamma,\beta) = e\int_\gamma^\infty \frac{e^{-\nu}d\nu}{\sqrt{\beta^2+\nu^2}}$$

$\alpha = \left(\frac{z}{a}\right)$, $\beta = \left(\frac{r}{a}\right)$, $\gamma = \left(\frac{z+l'}{a}\right)$

Equation (30) is the steady-state final solution.

The function I(x,y) which is defined by

$$I(x,y) = e^x \int_x^\infty \frac{e^{-\nu} d\nu}{\sqrt{y^2+\nu^2}} \tag{30a}$$

can be approximated as follows:

a) When $y < 0.01x$, the value of y integrand can be neglected in comparison to that of ν. Consequently, for $y < 0.01x$, the function is approximated by

$$I(x,y) \approx e^x \int_x^\infty \frac{e^{-\nu} d\nu}{\nu}$$

$$\approx e^x \, W(x) \tag{30b}$$

where W(x) is the exponential integral commonly termed the well function in Hydrology.

b) When $x > 10y$. The function I(x,y) can be written as follows:

$$I(x,y) = e^x \left[\int_0^\infty \frac{e^{-\nu} d\nu}{\sqrt{y^2+\nu^2}} - \int_0^x \frac{e^{-\nu} d\nu}{\sqrt{y^2+\nu^2}} \right] \tag{30c}$$

When $x > 10y$, the value of ν in the radical of the integrand of the second integral can be neglected in comparison with y. Consequently,

$$I(x,y) \approx e^x \left[H(y) + \frac{e^{-x}-1}{y} \right]$$

$$\approx e^x H(y) + \frac{1-e^x}{y} \tag{30d}$$

Where H(y) is the first of the integrals of equation (30) which is defined (Erdlyi, 1954) by

$$H(y) = \frac{\pi}{2} [H_0(y) - Y_0(y)] \tag{30e}$$

where

$H_0(\beta)$ is the zero order Struve function

$Y_0(\beta)$ is the zero order Bessel function of the second kind.

(c) Approximations of the steady-state solution

When $\beta < .01\alpha$, equation (30), using (30b), can be approximated by

$$s = \frac{Q}{4\pi K l'} [c + 2e^{\alpha}W(\alpha) - 2e^{\gamma}W(\gamma)] \tag{31}$$

in which the terms have been defined.

When $\beta > 10\gamma$, equation (30, using (30c), can be approximated by

$$s = \frac{Q}{4\pi K l'} [c + 2 \{e^{\alpha}H(\beta) + \frac{1-e^{\alpha}}{\beta}\} - 2 \{e^{\gamma}H(\beta) + \frac{1-e^{\gamma}}{\beta}\}] \tag{32}$$

where $H(\beta)$ is a function defined by equation (30e).

Case 2. Flow to a spherical cavity, penetration l', constant discharge.

(a) Unsteady-state

The unsteady-state solution of flow to a well perforated throughout its depth of penetration, discussed above, was the result of averaging the flow to a spherical cavity over the length l'. Differentiating with respect to l' will therefore give the solution for the present case, if Q is replaced with ql'.

Differentiating equation (22), or the corresponding equation for $z > l'$, with respect to l', yields

$$\bar{s} = \frac{B}{2}\Big[\int_0^\infty \frac{e^{-(l'+z)\sqrt{p/\nu+\bar{\alpha}^2}}\bar{\alpha}J_0(\bar{\alpha}r)\,d\bar{\alpha}}{p\sqrt{p/\nu+\bar{\alpha}^2}} + \int_0^\infty \frac{e^{-(l'-z)\sqrt{p/\nu+\bar{\alpha}^2}}\bar{\alpha}J_0(\bar{\alpha}r)\,d\bar{\alpha}}{p\sqrt{p/\nu+\bar{\alpha}^2}}$$

$$- 2\int_0^\infty \frac{\frac{1}{a}\, e^{-(l'+z)\sqrt{p/\nu+\bar{\alpha}^2}}\bar{\alpha}J_0(\bar{\alpha}r)\,d\bar{\alpha}}{p\sqrt{p/\nu+\bar{\alpha}^2}\left(\frac{1}{a}+\sqrt{p/\nu+\bar{\alpha}^2}\right)}\Big] \qquad (33)$$

where $B = \frac{q}{2\pi K}$, q being the discharge of the cavity.

Rearranging (33) and using the convolution and translation theorems

$$s = \frac{B}{2}\int_0^\infty \Big\{ 1*\Big(L^{-1}\Big[\frac{e^{-(l'+z)E}}{\sqrt{p}}\Big] + L^{-1}\Big[\frac{e^{-(l'-z)E}}{\sqrt{p}}\Big]$$

$$- \frac{2}{a}\, L^{-1}\Big[\frac{e^{-(l'+z)E}}{\sqrt{p}\left(\frac{\sqrt{\nu}}{a}+\sqrt{p}\right)}\Big]\Big)\Big\}\,\bar{\alpha}J_0(\bar{\alpha}r)\,d\bar{\alpha} \qquad (34)$$

Using Appendix II, equations (3), (5), and (6), equation (34) reduces to

$$s = \frac{B}{2}\int_0^t \Big\{ \frac{1}{2\sqrt{\nu\pi}}\,\frac{e^{-\left(\frac{r^2+(l'+z)^2}{4\nu\tau}\right)}}{\tau^{3/2}} + \frac{1}{2\sqrt{\nu\pi}}\,\frac{e^{-\left(\frac{r^2+(l'-z)^2}{4\nu\tau}\right)}}{\tau^{3/2}}$$

$$- \frac{1}{2a}\,\frac{e^{\frac{r^2}{4\nu\tau}}}{\tau}\, e^{\frac{l'+z}{a}}\, e^{\frac{\nu\tau}{a^2}}\,\mathrm{erfc}\left(\frac{\sqrt{\nu\tau}}{a} + \frac{l'+z}{\sqrt{4\nu\tau}}\right)\Big\}\,d\tau \qquad (35)$$

Making the substitution $\lambda^2 = \left(\frac{r^2 + (l'+z)^2}{4\nu\tau}\right)$ in the first two integrals of equation (35) and making the substitution $y = \frac{r^2}{4\nu\tau}$ in the last integral, equation (35) reduces to:

$$s = \frac{q}{4\pi K}\Big[\frac{1}{R}\,\mathrm{erfc}(U) + \frac{1}{R'}\,\mathrm{erfc}(U')$$

$$- \frac{1}{a}\int_u^\infty \frac{e^{-y}}{y}\, e^{\frac{1'+z}{a}}\, e^{\frac{r^2}{4a^2 y}}\,\mathrm{erfc}\left(\frac{r}{2a\sqrt{y}} + \frac{(1'+z)\sqrt{y}}{r}\right)\Big]\,dy \qquad (36)$$

in which $U = \frac{R}{\sqrt{4\nu\tau}}$, $U' = \frac{R'}{\sqrt{4\nu\tau}}$, $R = \sqrt{r^2+(1+z)^2}$, $R' = \sqrt{r^2+(1-z)^2}$.

Equation (36) is the final unsteady-state solution.

(b) Steady-state solution

The steady-state case is obtained from (36) as $t \to \infty$. As $t \to \infty$, $\mathrm{erf}(\infty) = 1$ and the third term becomes

$$s = \frac{B}{2}\int_0^\infty \Big[\, e^{-(1'+z)\bar{\alpha}} + e^{-(1'-z)\bar{\alpha}} - \frac{2\,\frac{1}{a}\, e^{-(1'+z)\bar{\alpha}}}{\frac{1}{a} + \bar{\alpha}}\,\Big]\, J_0(\bar{\alpha} r)\, d\bar{\alpha} \qquad (37)$$

It can be shown from the solutions of completely perforated wells that the integral of (37) is equal to $I(\gamma,\beta)$. Consequently, the steady solution becomes

$$s = \frac{q}{4\pi K}\Big[\,\frac{1}{R} + \frac{1}{R'} - \frac{2}{a}\, I(\gamma,\beta)\,\Big] \qquad (38)$$

(c) Approximations of the steady-state solution

When $\beta < .01\gamma$, equation (38), can be approximated by

$$s = \frac{q}{4\pi K}\Big[\,\frac{1}{R} + \frac{1}{R'} - \frac{2}{a}\, e^{\gamma} W(\gamma)\,\Big] \qquad (39)$$

When $\beta > 10\gamma$, equation (38) can be approximated by

$$s = \frac{q}{4\pi K}\Big[\,\frac{1}{\sqrt{r^2+(1'+z)^2}} + \frac{1}{\sqrt{r^2+(1'-z)^2}} - \frac{2}{a}\Big\{\, e^{\gamma} H(\beta) + \frac{1-e^{\gamma}}{\beta}\,\Big\}\,\Big] \qquad (40)$$

Case 3. Flow to a hemispherical cavity, zero penetration, constant discharge. The solutions to this case are obtained from those of the spherical cavity with l' equal zero.

(a) Unsteady-state

The solution to this case, obtained from equation (36) with l' equal to zero, yields

$$s = \frac{q}{2\pi K}\left[\frac{1}{R_o}\,\mathrm{erfc}\left(\frac{R_o}{\sqrt{4\nu\tau}}\right) - \frac{1}{a}\int_u^\infty \frac{e^{-y}}{y}\, e^{\frac{z}{a}}\, e^{\frac{r^2}{4a^2y}}\,\mathrm{erfc}\left(\frac{r}{2a\sqrt{y}} + \frac{z\sqrt{y}}{r}\right) dy\right] \tag{41}$$

where $R_o = r^2 + z^2$.

(b) Steady-state solution

Equation (38) with l' equal to zero gives $s = \frac{q}{2\pi K}\left[\frac{1}{R_o} - \frac{1}{a} I(\alpha,\beta)\right]$. (42)

(c) Approximations of the steady-state solution

When $\beta < .01\alpha$, equation (42) can be approximated by

$$s = \frac{q}{2\pi K}\left[\frac{1}{R_o} - \frac{e^a}{a} W(\alpha)\right] \tag{43}$$

When $\beta > 10\alpha$, equation (42) can be approximated by

$$s = \frac{q}{2\pi K}\left[\frac{1}{R_o} - \frac{e}{a} H(\beta) + \frac{1-e^{-\alpha}}{a\beta}\right] \tag{44}$$

Case 4. Wells of finite depth, penetration $(l'-d)$ and constant discharge.

(a) Unsteady-state

$$s = [\, s(l') - s(d) \,] \tag{45}$$

where $s(l')$ is equation (26), the drawdown equation for complete perforation l' and $s(d)$ is the same equation with d replacing l'.

(b) Steady-state

$$s = [\ M(l') - M(d)\ ]$$

where $M(l')$ is equation (30) and $M(d)$ is equation (30) with l' replaced by d.

Discussion

The zero-order Hankel transform is used with respect to r in solving the differential equation

$$H_0 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) \right] = -\alpha^2 F(\alpha) + \left[\left(r \frac{\partial f}{\partial r} \right) J_0(\alpha r) + (\alpha r) \; J_1(\alpha r) f(r) \right]_{r=0}^{r=\infty}$$

The use of the transform is justified provided the two terms in the right-hand brackets approach zero or a finite value when evaluated at the limits, that is, as r approaches infinity and as it approaches zero. Taking the limit as r approaches zero, we see that the first term in brackets is equation (14), or the flow condition on the inner boundary. The second term in brackets approaches zero as r approaches zero because J_1 is bounded. Taking the limit of the first term in brackets as r approaches infinity, we see that it approaches zero as $r^{-3/2}$, for $r \frac{\partial f}{\partial r}$ acts as $\frac{1}{r}$ and $J_0(\alpha r)$ acts as $r^{-1/2}$ for large r. The last term in brackets also behaves as $r^{-3/2}$ for large values of r since $J_1(\alpha r)$ behaves as $r^{-1/2}$ and f(r) behaves as $\frac{1}{r}$. Since the solution of the differential equation checks graphically, and since for large r the potential approaches hemispherical potential which varies as $\frac{1}{r}$ for large r, the use of the Hankel transform is justified.

The final-value theorem is used to arrive at the steady-state solutions for the various cases. It can be stated

$$\lim_{p \to 0} pf(p) = \lim_{t \to \infty} f(t)$$

This theorem holds provided the function treated is analytic along the axis of the imaginaries and in the right-hand half plane.

Recalling equation (22),

$$\bar{s} = \frac{A}{2}\int_0^\infty \frac{(2-e^{-(1+z)\sqrt{p/\nu+\bar{\alpha}^2}}-e^{-(1-z)\sqrt{p/\nu+\bar{\alpha}^2}})\bar{\alpha}J_0(\bar{\alpha}r)\,d\bar{\alpha}}{p(p/\nu + \bar{\alpha}^2)}$$

$$- A\int_0^\infty \frac{\frac{1}{\bar{a}}(e^{-z\sqrt{p/\nu+\bar{\alpha}^2}}-e^{-(1+z)\sqrt{p/\nu+\alpha^2}})\alpha J_0(\alpha r)\,d\alpha}{p(p/\nu+\bar{\alpha}^2)\ (\frac{1}{a} + \sqrt{p/\nu+\bar{\alpha}^2})}$$

we observe that both integrals have negative exponentials and J_0 terms in the numerator which allow the integral to converge between the limits of zero and infinity after we take $\lim_{p\to 0} pf(p)$. Now $p(p/\nu+\alpha^2)$ is analytic in the right-hand half plane and on the axis of imaginaries. Moreover, $(\frac{1}{a} + \sqrt{p/\nu+\bar{\alpha}^2})$ behaves as

$$\int_0^\infty \frac{d\bar{\alpha}}{\frac{1}{a} + \sqrt{\frac{1}{a^2}}}$$

and is also analytic on the axis of imaginaries and in the right-hand half plane. If the components of an integral are analytic, the integral is analytic, and we can conclude that the use of the final-value theorem is justified.

Tables 1 and 2, graphs 1 through 15, and Figure 2, are all for the steady-state, totally perforated case.

Table 1 contains the tabulation of the function $H(\beta)$ which is contained in the approximate steady-state solution for a totally perforated, partially penetrating well.

Table 2 contains the tabulation of the fully perforated, steady-state case. These are shown graphically in Graphs 2 through 5.

Figure 1 depicts the four cases analyzed in this thesis. Each case is treated individually and no attempt is made to analyze more than one case at a time.

Graphs 1 and 2 show the steady-state drawdown distribution for various horizontal sections taken through the aquifer. This technique of horizontal sections over a rather large vertical span was used to decrease the amount of computer time needed. It should be pointed out that the drawdown is greatest near the half-length of penetration.

Figure 2 utilizes the data compiled in Table 2 and is used as an example for a value of $l'/a = 1$. The equipotential lines intersect the leaky bed at an oblique angle, which was expected from the tangent law of refraction. The dashed portion of the equipotential graph indicates data that had to be interpolated.

Graph 3 again illustrates drawdown taken at various points in a horizontal plane. Unlike graphs 1 and 2, graph 3 is displayed on plain coordinates.

Graphs 4 through 15 show the effects of the two integrals upon the steady-state solution for various fixed values of z/a and l'/a.

Conclusions

The differential equation is solved by two techniques. Both solutions reduce to Saad's [1960] solution when the leakage term is allowed to go to zero. Thus it is reasonably assured that the solutions presented in this thesis are correct within the stated assumptions.

It is noted that a potential singularity varying as r^{-1} or z^{-1} exists at $r = 0 = z$. We can, however, substitute the reciprocal radius of the well, r_w^{-1}, for r^{-1} and obtain a useful solution for the potential at the well. This solution holds to a very high degree as a close approximation is made in discounting the leakage over πr^2_w. Since the leakage over this area is extremely small in comparison to the total flow, it can be considered negligible.

An estimate of the drawdown for a finite cylindrical well can be made by extrapolating the potential distribution inward from the smallest radius for which it was calculated, using the average potential over the cylindrical surface corresponding to that smallest radius.

Suggestions for Future Work

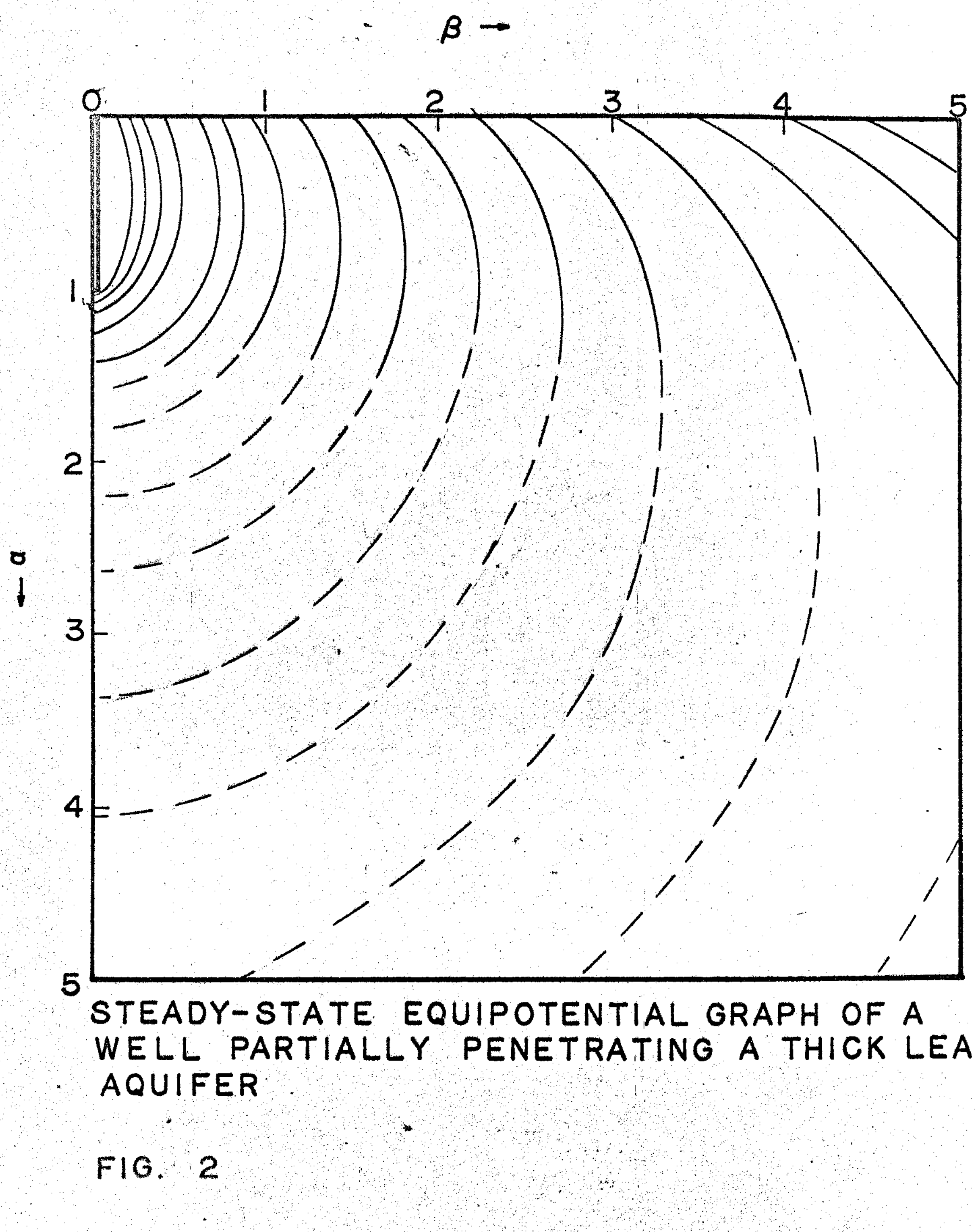
The unsteady-state equation representing flow to a well of penetration and perforation l' would prove very valuable if it were tabulated. This solution should be programmed and tabulated in a computer.

Also, the steady-state solution could be tabulated more extensively, in the following fashion:

$$\frac{s_w}{Q/4\pi K l'} = f\left(\frac{l'}{a}, \frac{r_w}{l'}\right) \quad \text{or} \quad f\left(\frac{l'}{a}, \frac{r_w}{a}\right).$$

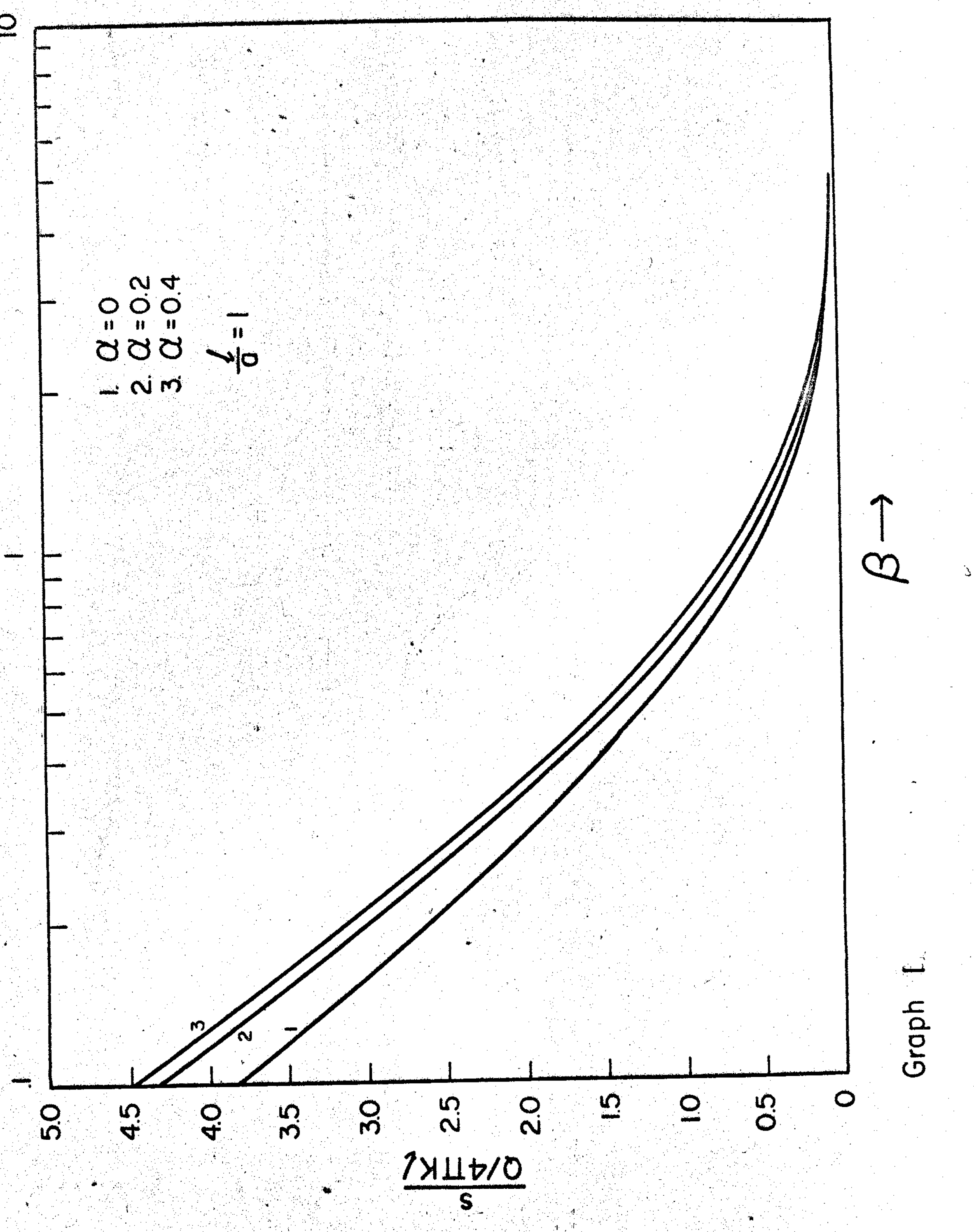
This would greatly enhance the practical usefulness of the solution.

Figure 2, showing the potential distribution around a well, does not include stream lines. The flow is axially symmetric or "axisymmetric." The Stokes stream function could be obtained by numerical integration using digital equipment, though this might be needlessly time-consuming. However, the streamlines can be obtained to a good approximation by graphical construction.

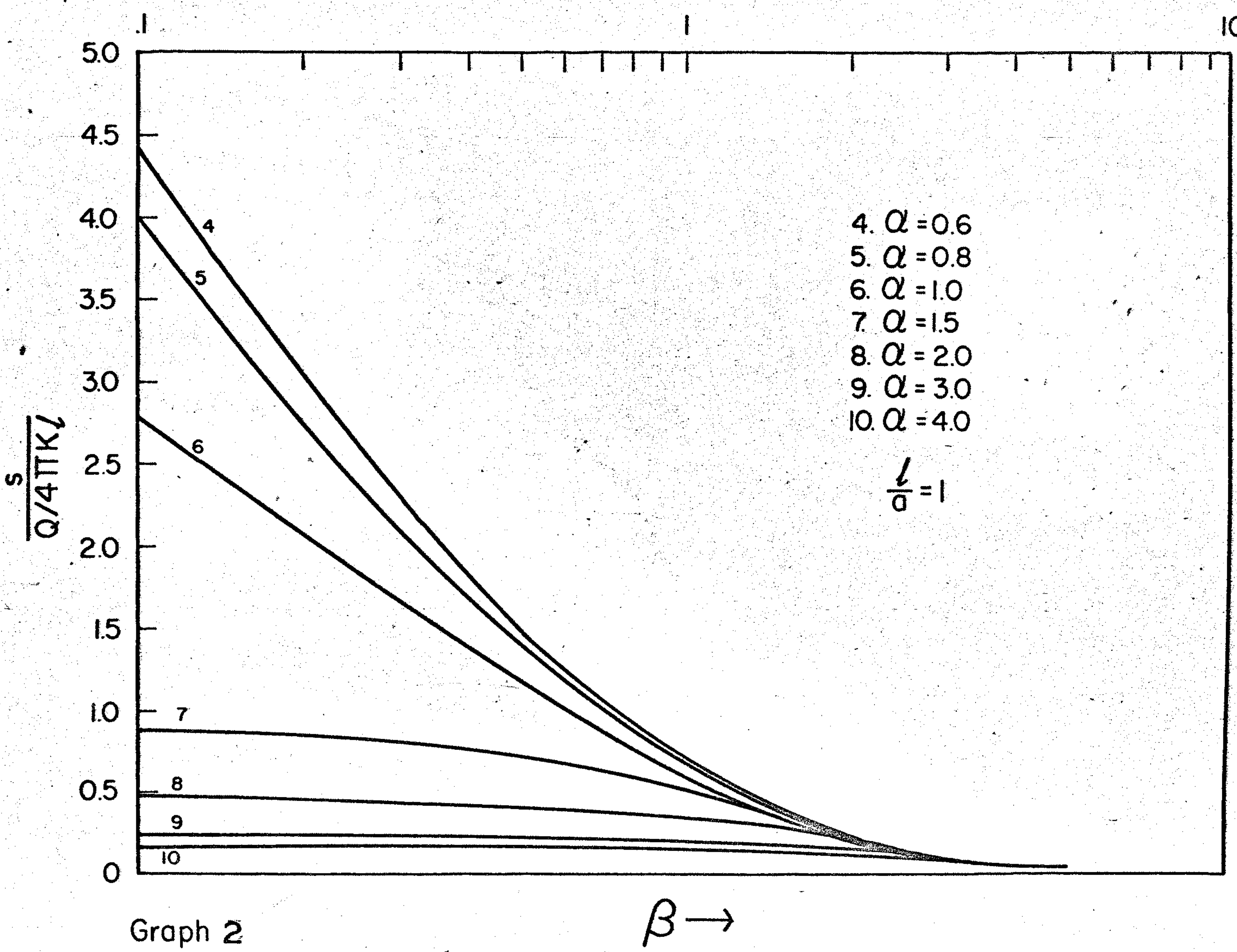


STEADY-STATE EQUIPOTENTIAL GRAPH OF A WELL PARTIALLY PENETRATING A THICK LEAKY AQUIFER

FIG. 2

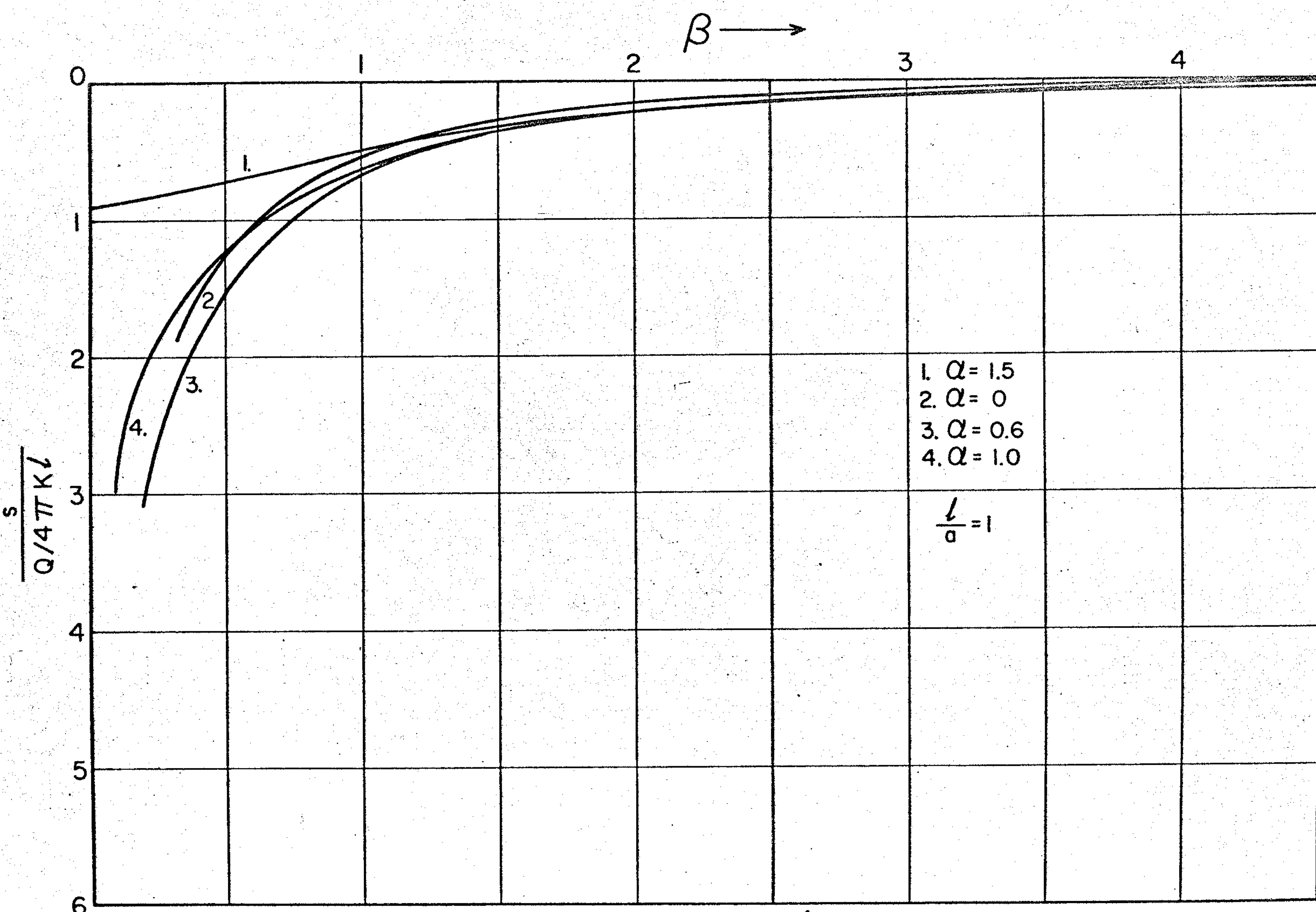


Graph I



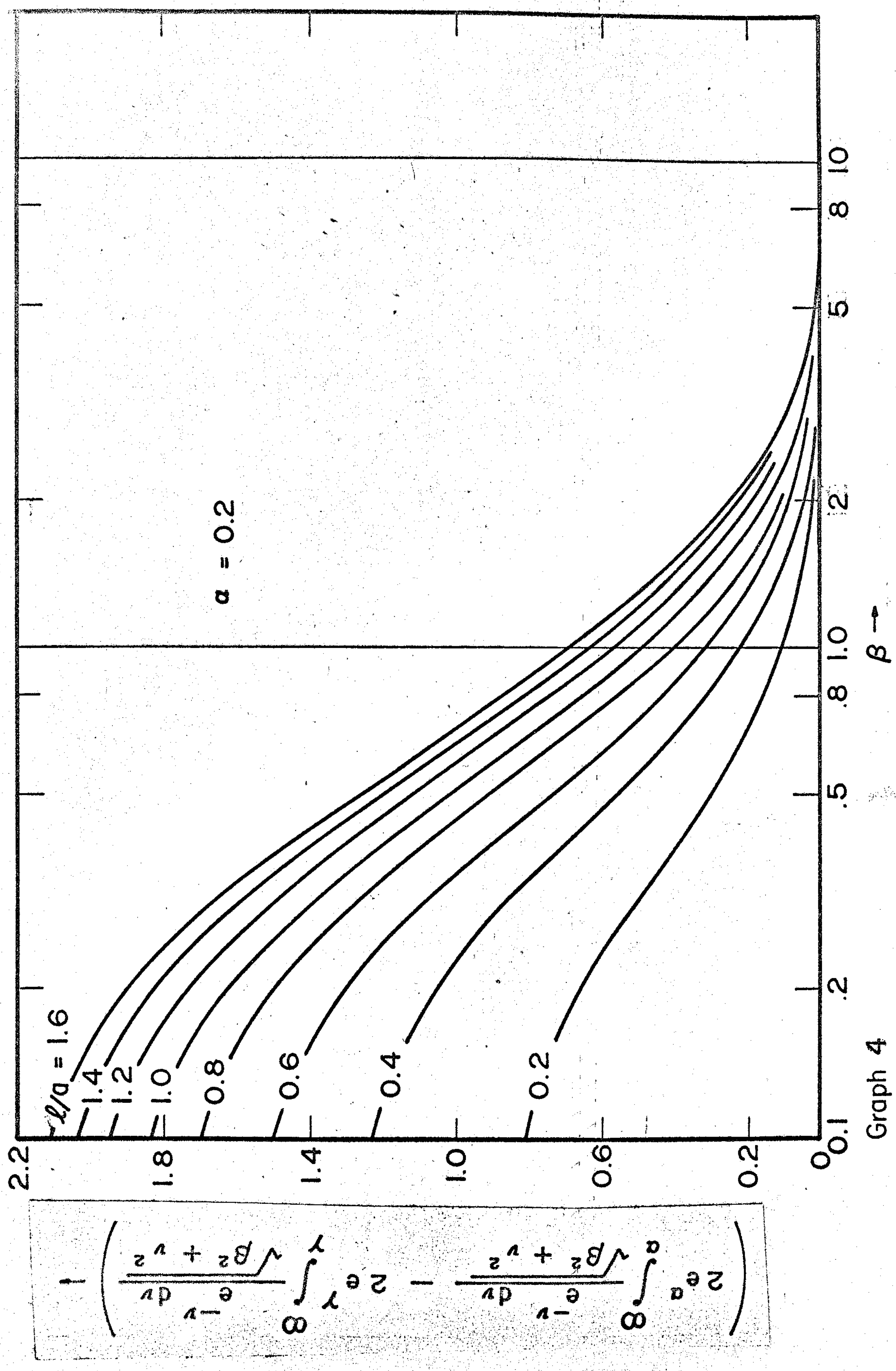
.1
1
10
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
s / Q/4πKl
4
5
6
7
8
9
10
4. α = 0.6
5. α = 0.8
6. α = 1.0
7. α = 1.5
8. α = 2.0
9. α = 3.0
10. α = 4.0
l/a = 1
β →

Graph 2



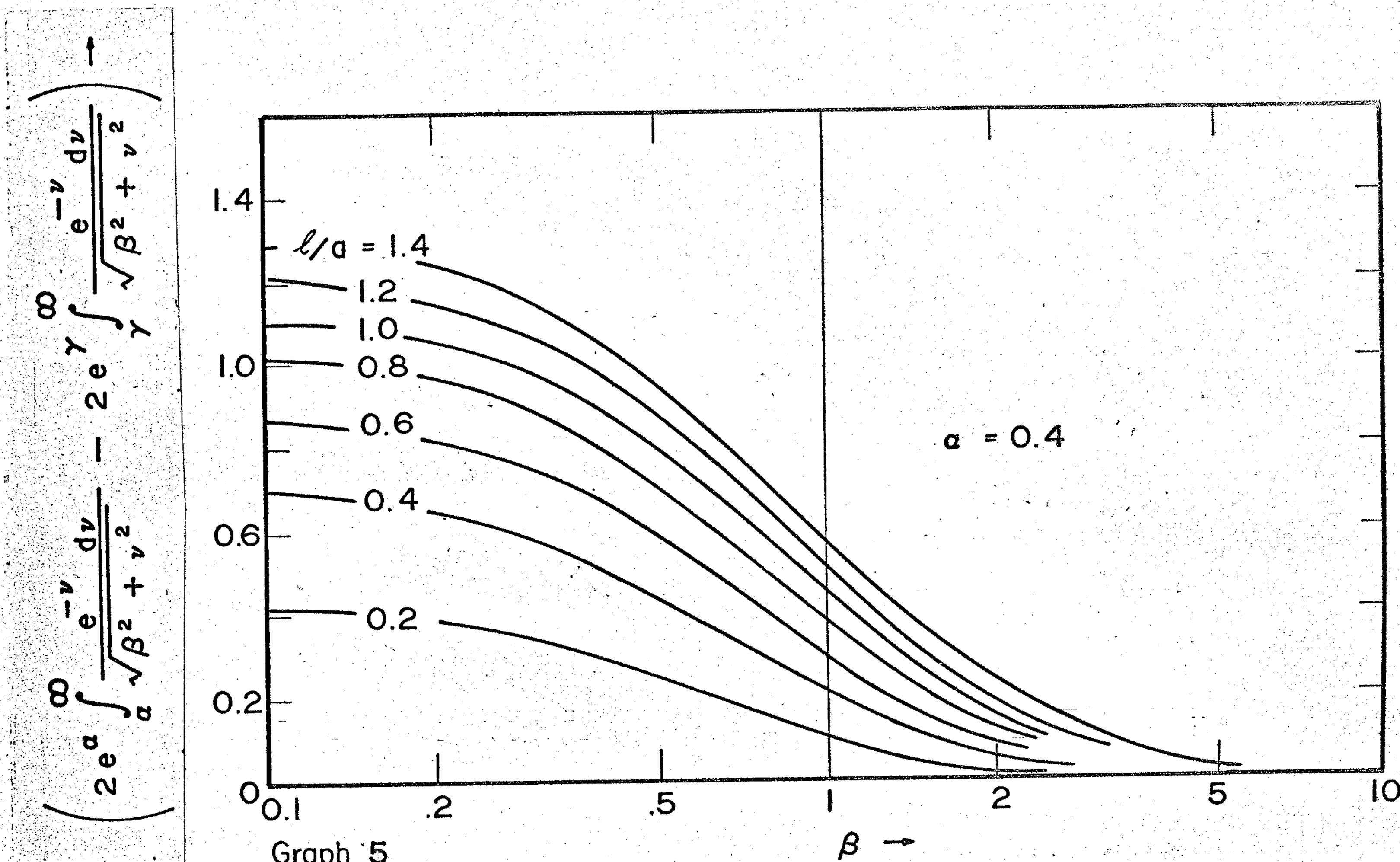
STEADY-STATE DRAWDOWN CURVES, $l/a = 1$, α AS INDICATED

Graph 3



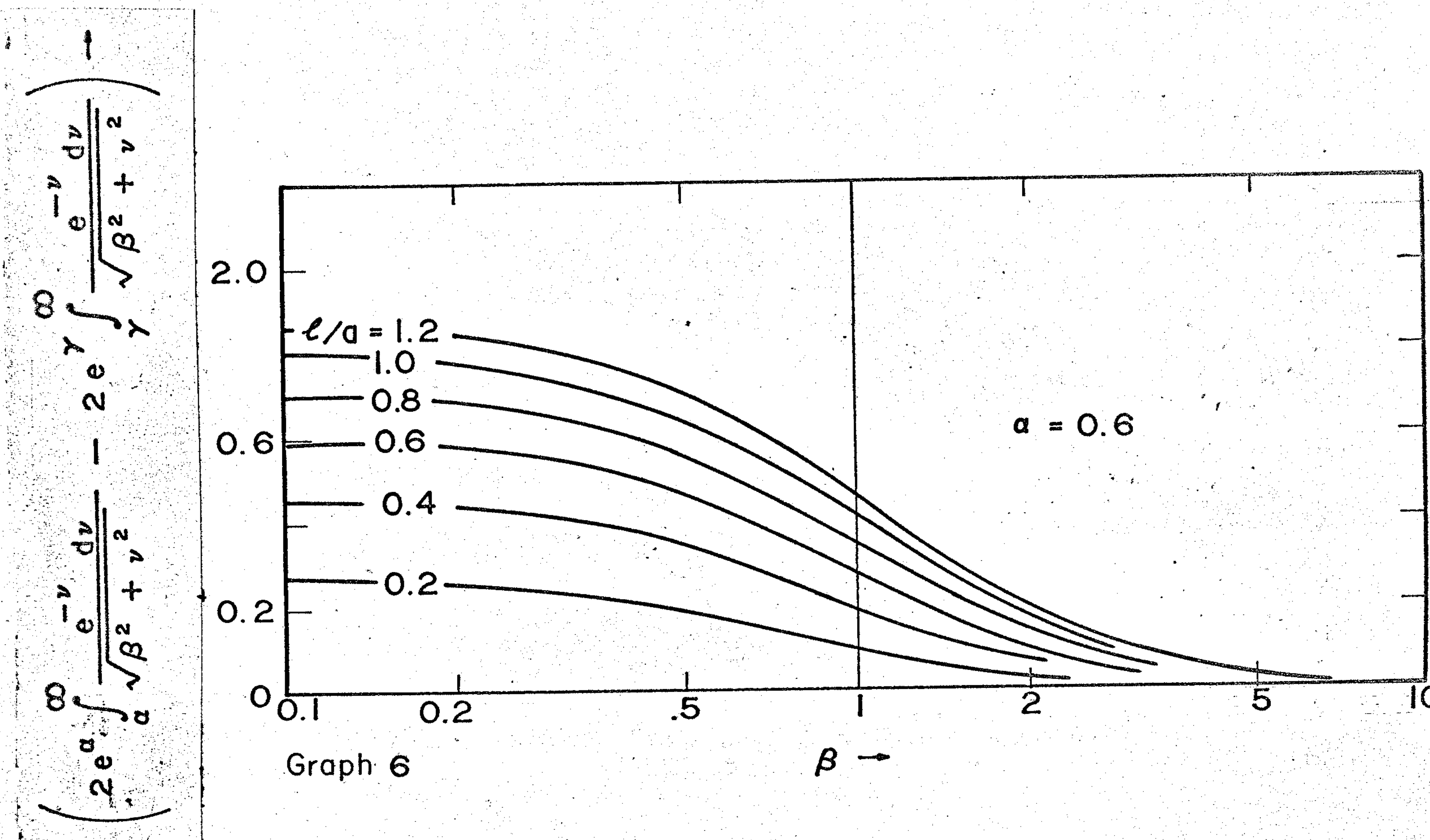
α = 0.2
β →
ℓ/a = 1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.1
.2
.5
.8
1.0
2
5
8
10
2.2
1.8
1.4
1.0
0.6
0.2
0
$\left(2e^{\alpha}\int_{\alpha}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}} - 2e^{\gamma}\int_{\gamma}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}}\right) \to$

Graph 4

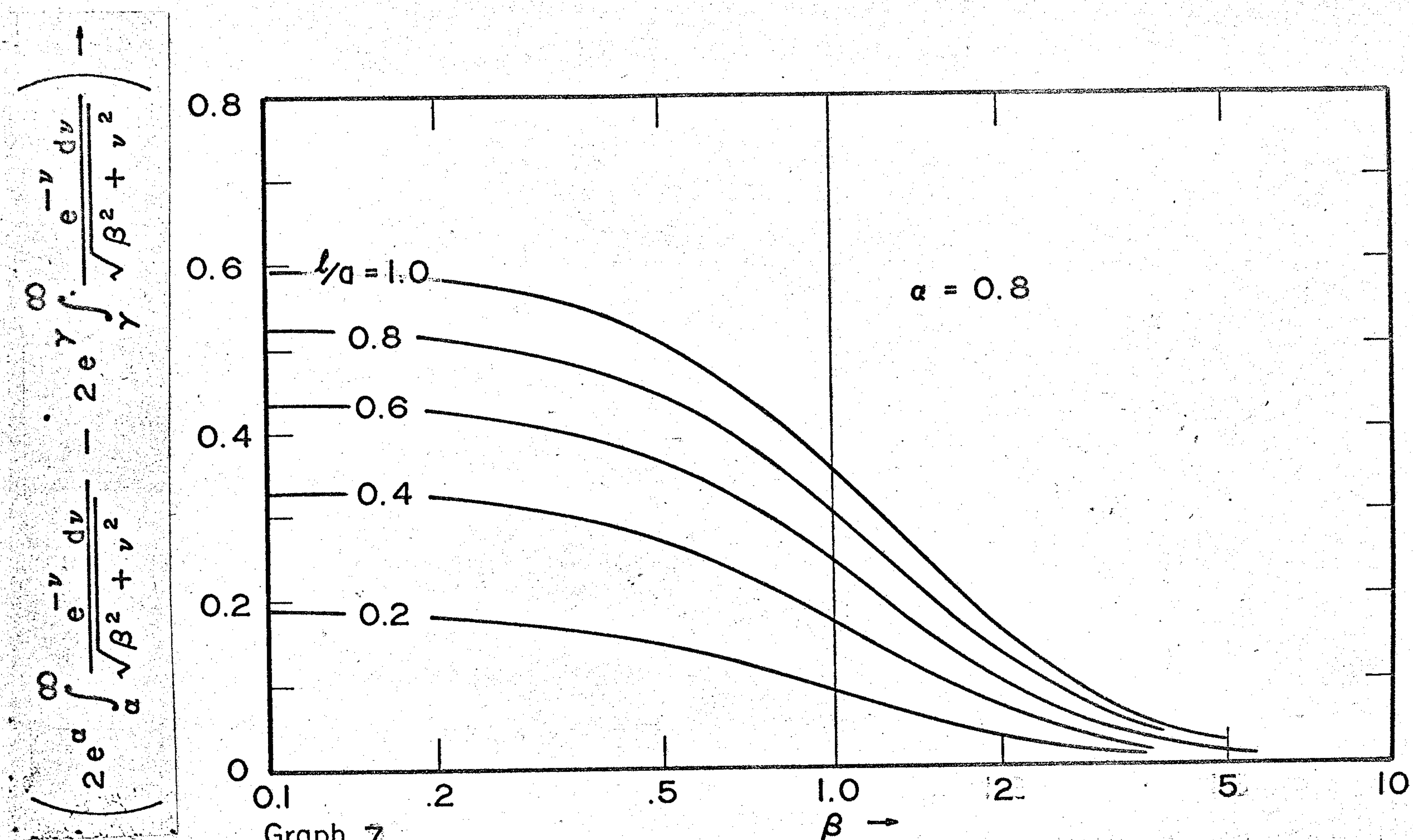


$\left(2e^{\alpha}\int_{\alpha}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}} - 2e^{\gamma}\int_{\gamma}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}}\right) \rightarrow$
1.4
1.0
0.6
0.2
0
ℓ/a = 1.4
1.2
1.0
0.8
0.6
0.4
0.2
α = 0.4
0.1
.2
.5
1
2
5
10
β →

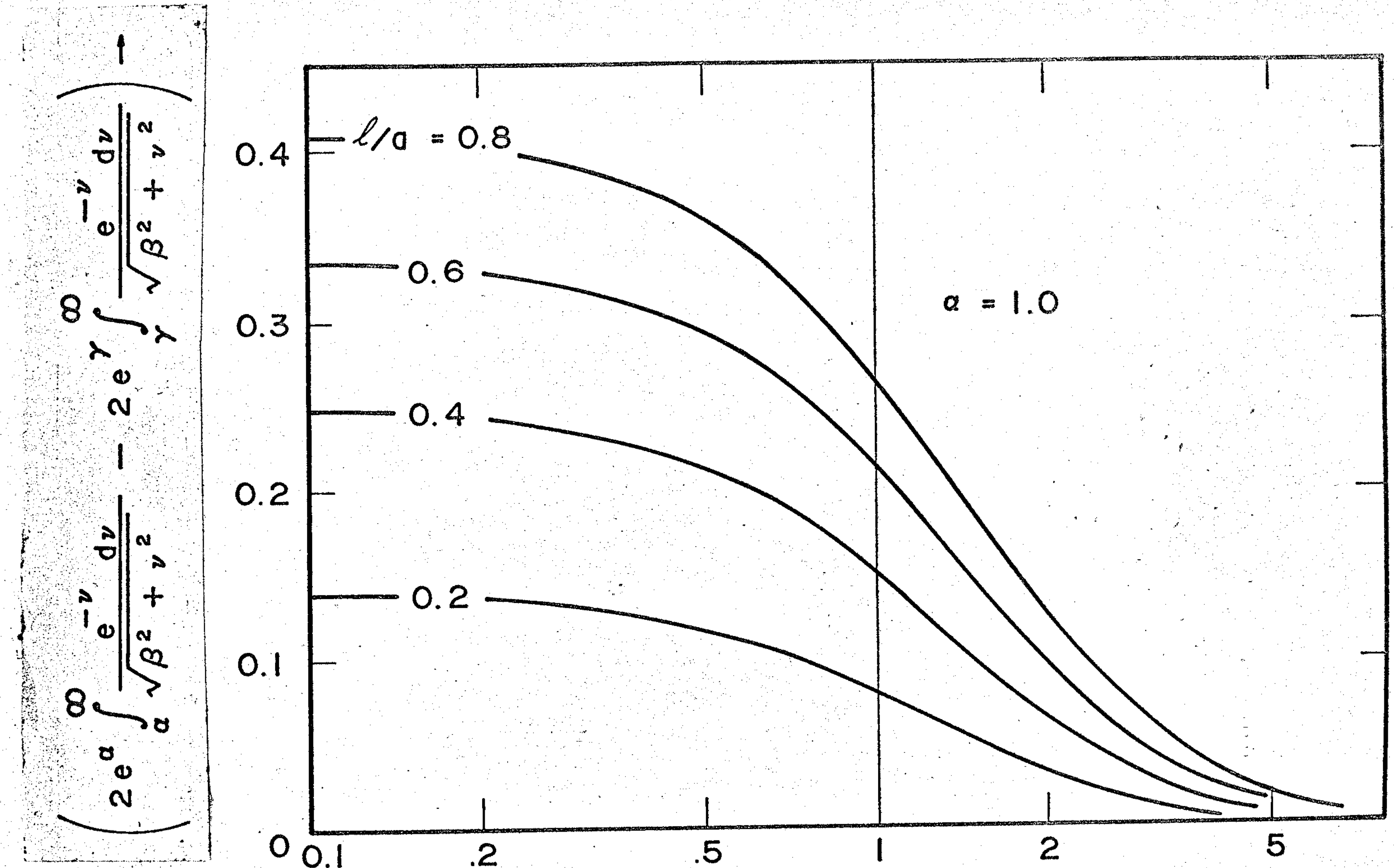
Graph 5


$\left(2e^{\alpha}\int_{\alpha}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}} - 2e^{\gamma}\int_{\gamma}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}}\right) \rightarrow$
2.0
0.6
0.2
0
ℓ/a = 1.2
1.0
0.8
0.6
0.4
0.2
α = 0.6
0.1
0.2
.5
1
2
5
10
β →

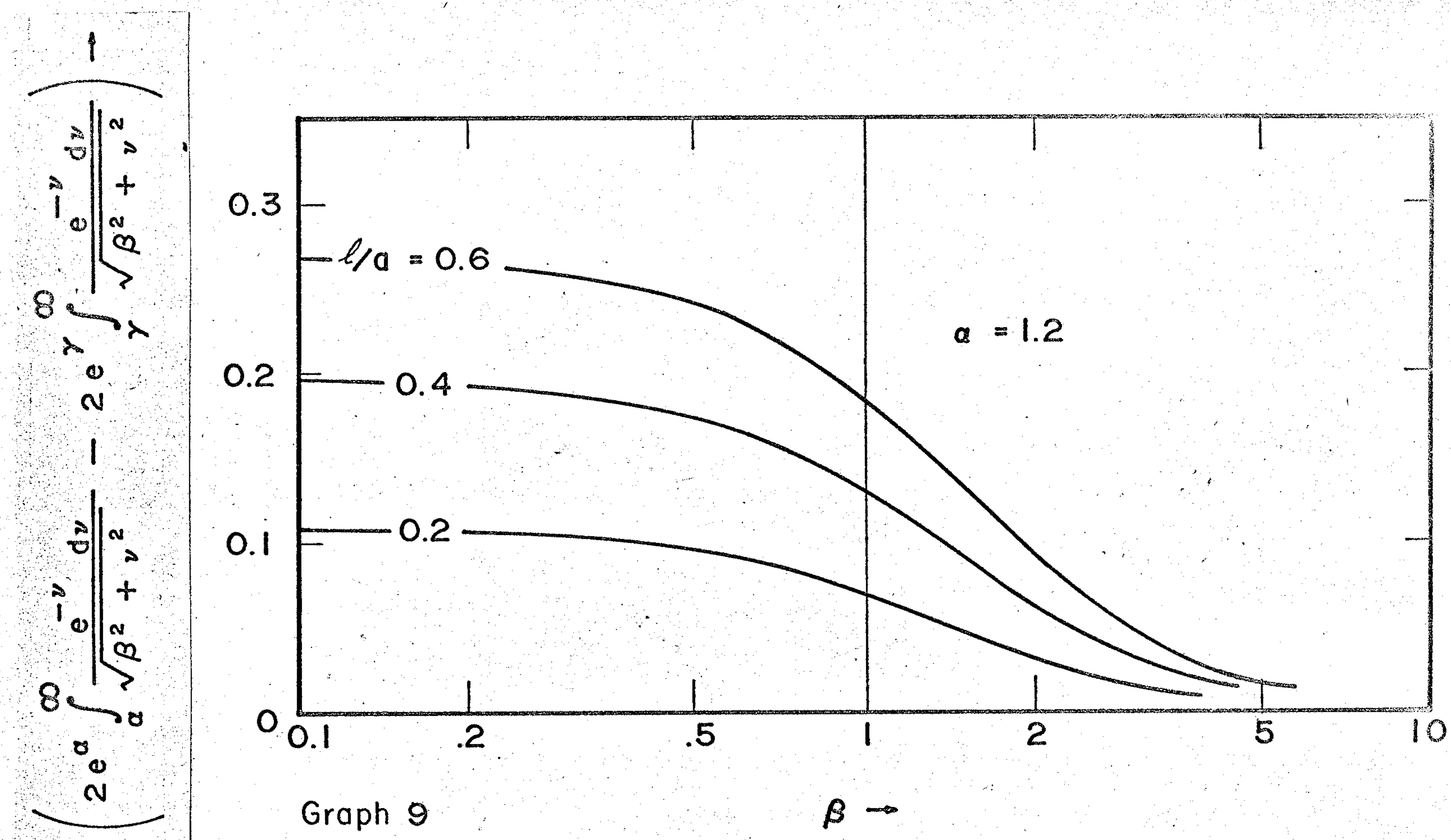
Graph 6


$\left(2e^{\alpha}\int_{\alpha}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}} - 2e^{\gamma}\int_{\gamma}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}}\right) \rightarrow$
0.8
0.6
0.4
0.2
0
ℓ/a = 1.0
0.8
0.6
0.4
0.2
α = 0.8
0.1
.2
.5
1.0
2
5
10
β →

Graph 7

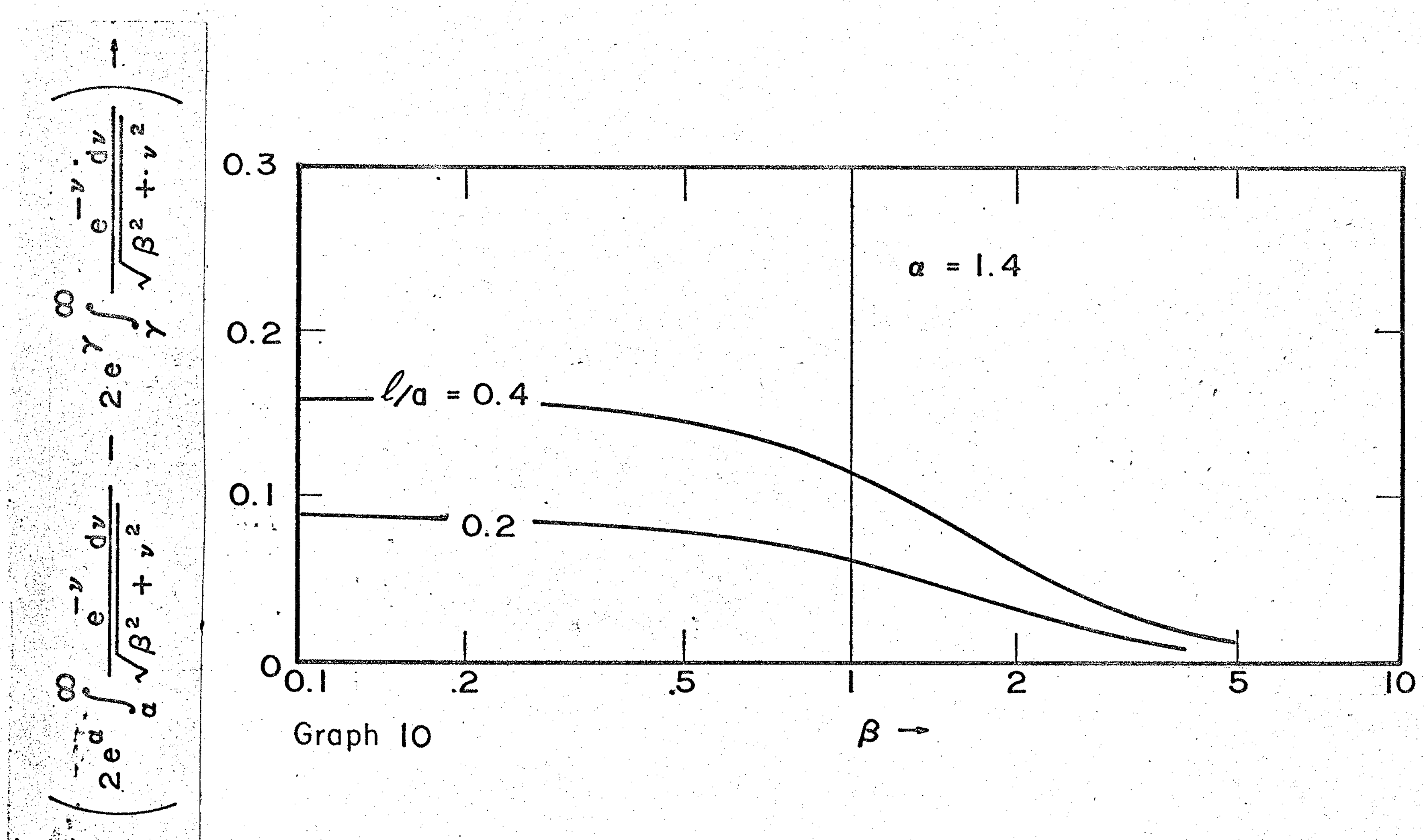

$\left(2e^{\alpha} \int_{\alpha}^{\infty} \frac{e^{-\nu}\, d\nu}{\sqrt{\beta^2 + \nu^2}} - 2e^{\gamma} \int_{\gamma}^{\infty} \frac{e^{-\nu}\, d\nu}{\sqrt{\beta^2 + \nu^2}} \right) \rightarrow$
0.4
0.3
0.2
0.1
0
0.1
.2
.5
1
2
5
β →
ℓ/a = 0.8
0.6
0.4
0.2
α = 1.0

Graph 8

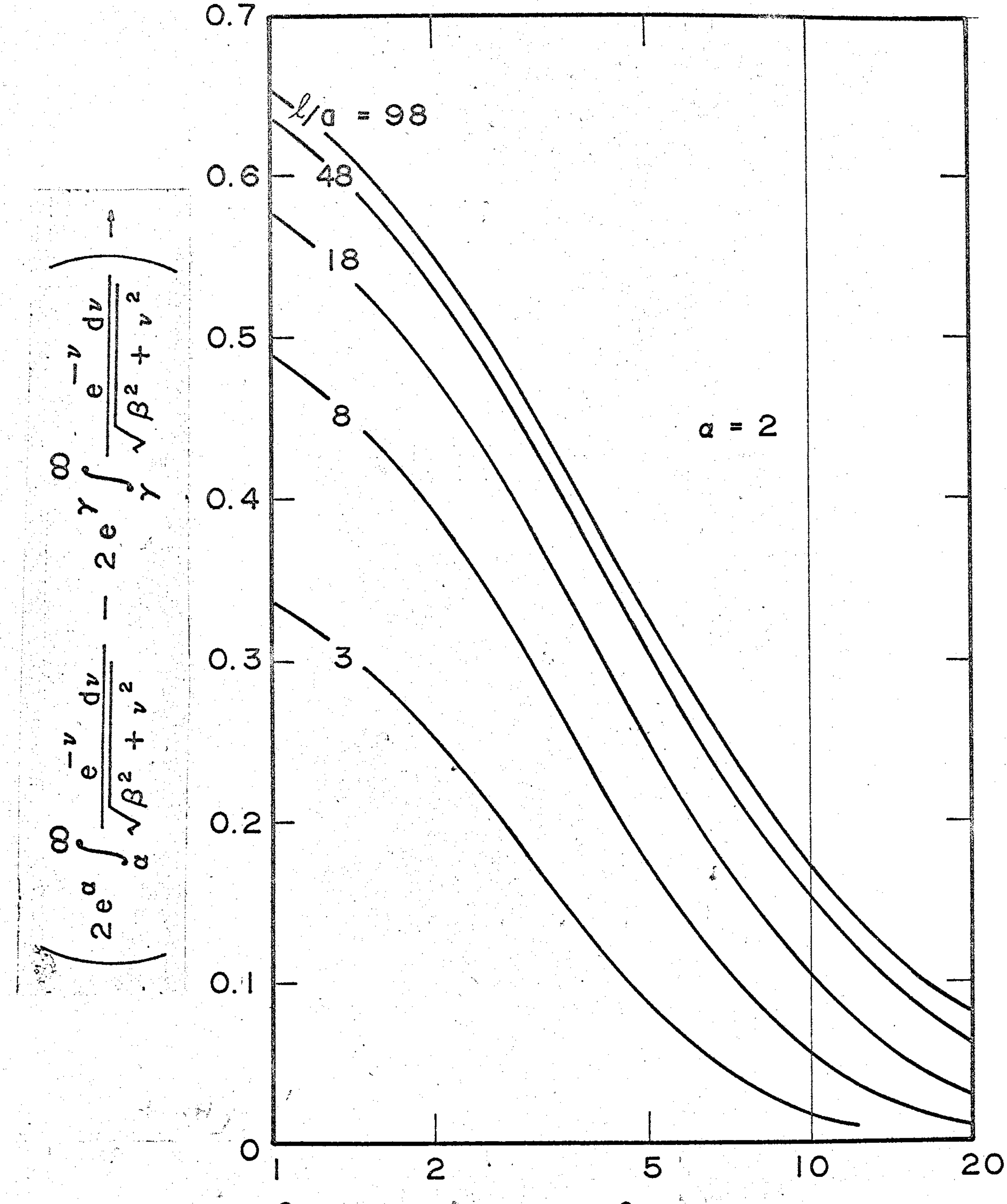


$\left(2e^{\alpha}\int_{\alpha}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}} - 2e^{\gamma}\int_{\gamma}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}}\right) \rightarrow$
0.3
0.2
0.1
0
ℓ/a = 0.6
0.4
0.2
α = 1.2
0.1
.2
.5
1
2
5
10
$\beta \rightarrow$

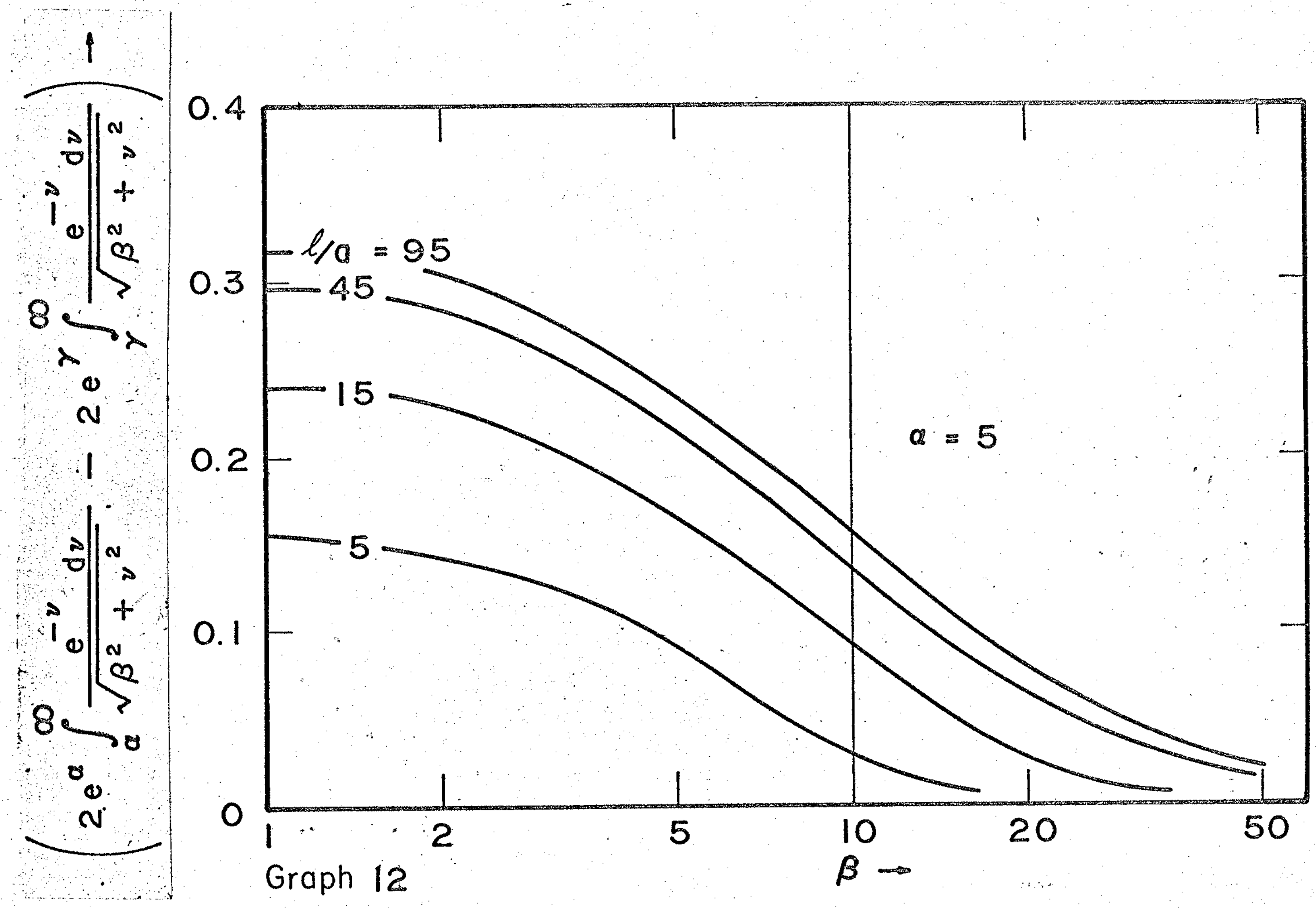
Graph 9


$\left(2e^{\alpha}\int_{\alpha}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}} - 2e^{\gamma}\int_{\gamma}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}}\right) \rightarrow$
0.3
0.2
0.1
0
0.1
.2
.5
1
2
5
10
α = 1.4
ℓ/a = 0.4
0.2
$\beta \rightarrow$

Graph 10

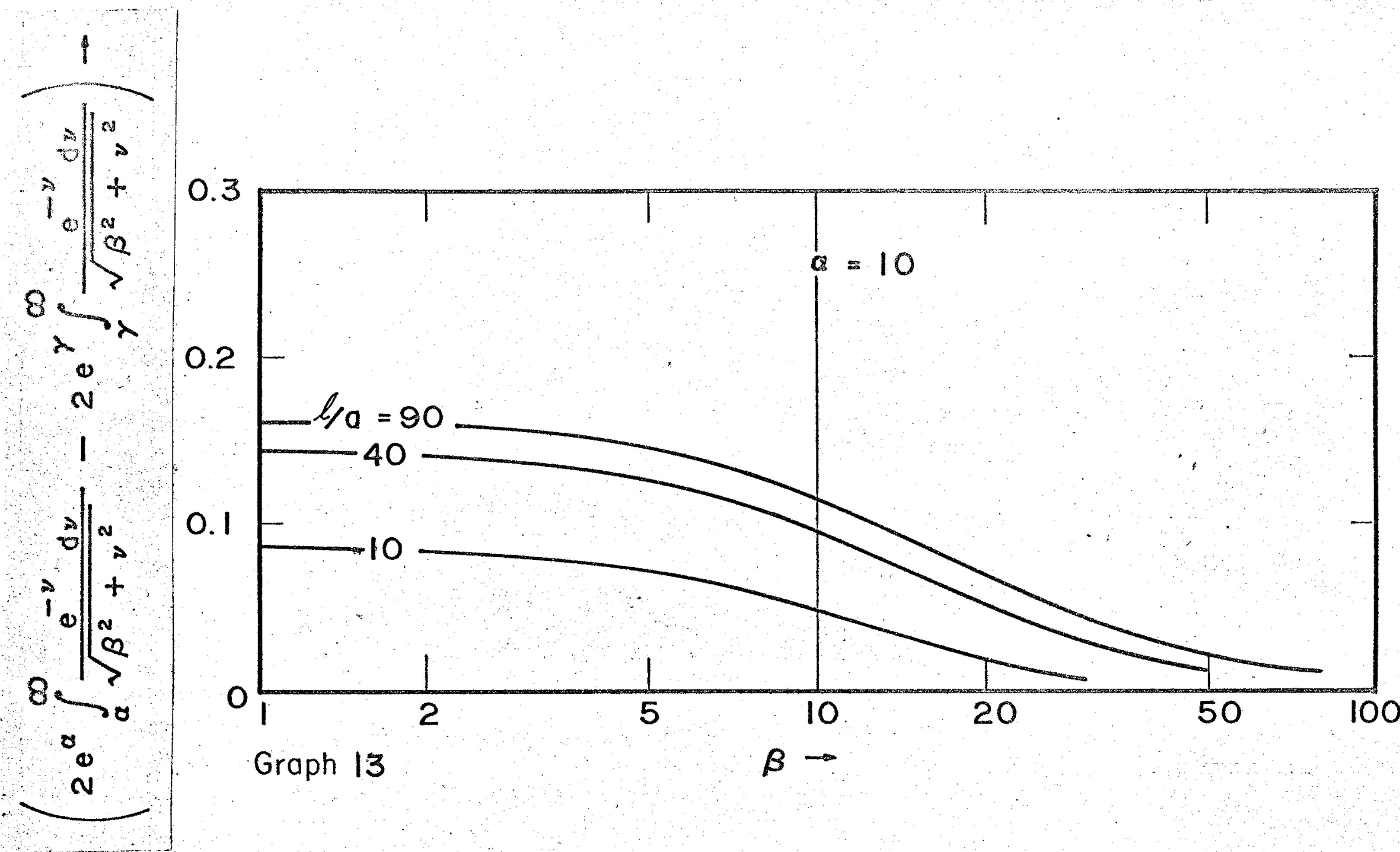

$\left(2e^{\alpha} \int_{\alpha}^{\infty} \frac{e^{-\nu}\, d\nu}{\sqrt{\beta^2 + \nu^2}} - 2e^{\gamma} \int_{\gamma}^{\infty} \frac{e^{-\nu}\, d\nu}{\sqrt{\beta^2 + \nu^2}} \right) \rightarrow$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
ℓ/a = 98
48
18
8
3
a = 2
1
2
5
10
20
β →

Graph IIb



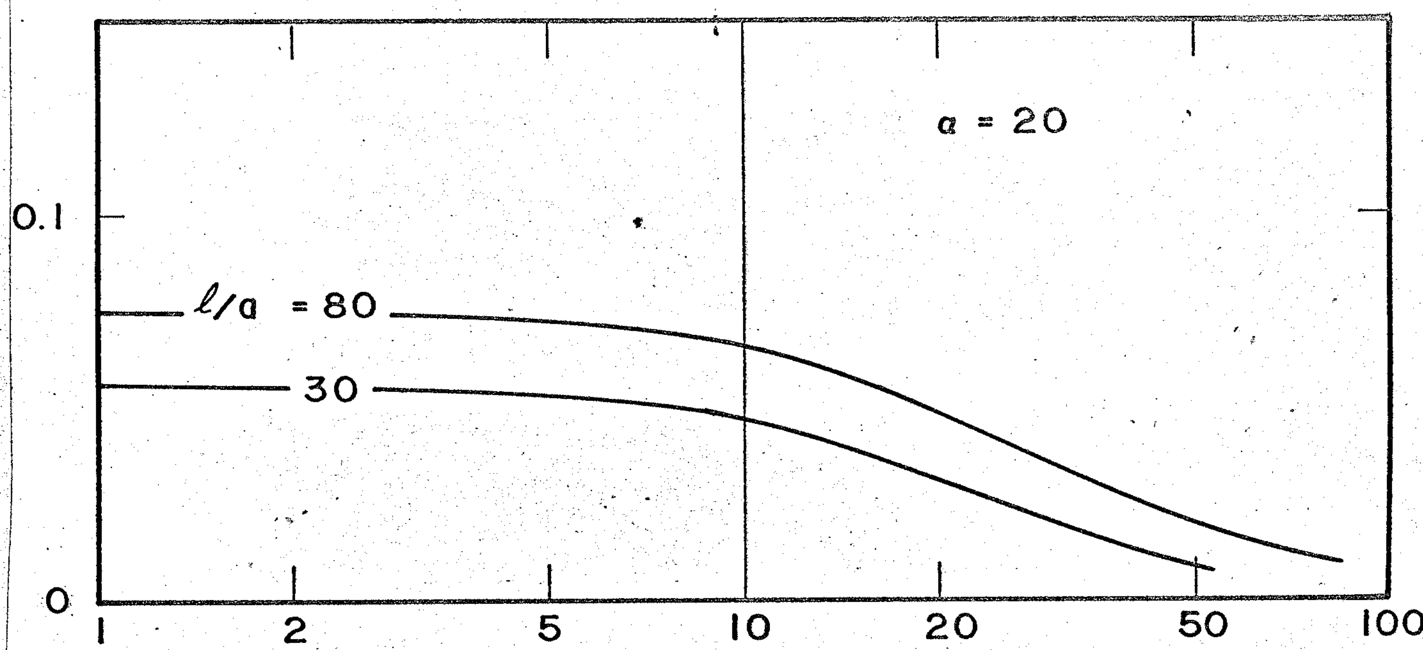
$\left(2e^{\alpha}\int_{\alpha}^{\infty}\frac{e^{-v}\,dv}{\sqrt{\beta^2+v^2}} - 2e^{\gamma}\int_{\gamma}^{\infty}\frac{e^{-v}\,dv}{\sqrt{\beta^2+v^2}}\right) \rightarrow$
0.4
0.3
0.2
0.1
0
ℓ/a = 95
45
15
5
α = 5
1
2
5
10
20
50
$\beta \rightarrow$

Graph 12



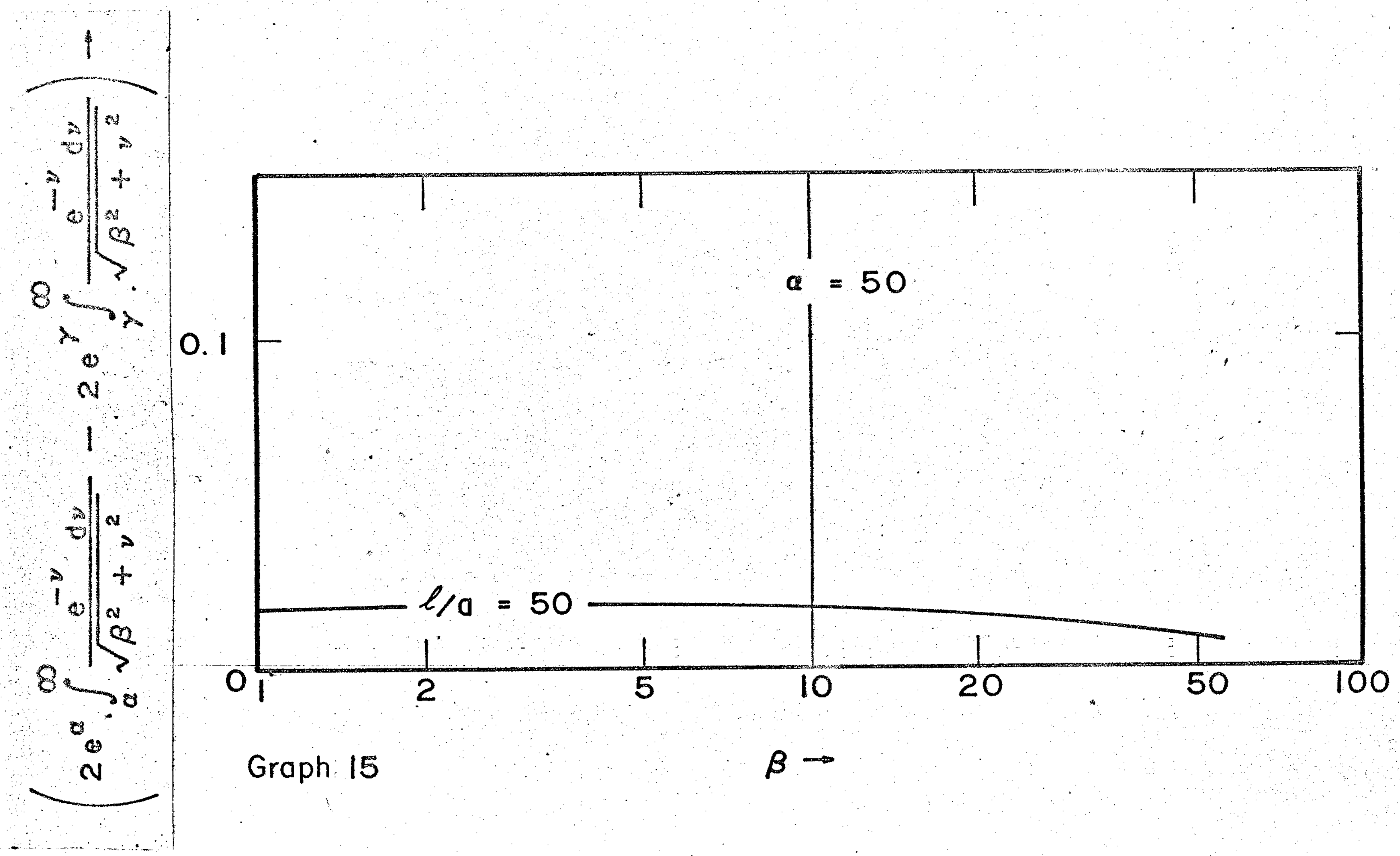
$\left(2e^{\alpha} \int_{\alpha}^{\infty} \frac{e^{-\nu}\, d\nu}{\sqrt{\beta^2 + \nu^2}} - 2e^{\gamma} \int_{\gamma}^{\infty} \frac{e^{-\nu}\, d\nu}{\sqrt{\beta^2 + \nu^2}} \right) \rightarrow$
0.3
0.2
0.1
0
$\alpha = 10$
$l/a = 90$
40
10
1
2
5
10
20
50
100
$\beta \rightarrow$

Graph 13



$$\left(2e^{\alpha}\int_{\alpha}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}} - 2e^{\gamma}\int_{\gamma}^{\infty}\frac{e^{-\nu}\,d\nu}{\sqrt{\beta^2+\nu^2}}\right) \rightarrow$$

Graph 14



$\left(2e^{\alpha} \int_{\alpha}^{\infty} \frac{e^{-\nu}\, d\nu}{\sqrt{\beta^2 + \nu^2}} - 2e^{\gamma} \int_{\gamma}^{\infty} \frac{e^{-\nu}\, d\nu}{\sqrt{\beta^2 + \nu^2}} \right) \rightarrow$
0.1
0
α = 50
ℓ/a = 50
1
2
5
10
20
50
100
β →

Graph 15

Symbols

$\frac{1}{a}$ = Leakage factor, $\frac{K'}{b'K}$ (L^{-1})

b' = thickness of semipervious layer (L)

d = portion of penetration length that is not perforated (L)

K = hydraulic conductivity of the aquifer LT^{-1}

K' = hydraulic conductivity of the semipervious bed LT^{-1}

l' = length of penetration of the well (L)

p = time t transformed with respect to the Laplace transform

Q = discharge (L^3)

q = discharge per unit area (L)

r = radial distance from the axis of the pumping well to any point in space (L)

s = drawdown at any point in the aquifer at any time t since pumping began (L)

$\bar{s}$ = s transformed with respect to the Laplace transform

$\bar{s}_c$ = $\bar{s}$ transformed with respect to the Fourier Cosine transform

Ss = specific storage, defined as the amount of water which a unit volume of the aquifer releases from storage under a unit decline in head (L^{-1})

$\bar{V}$ = the zero order Hankel transform of $\bar{s}$

$\bar{V}c$ = the zero order Hankel transform of $\bar{s}_c$

ω = z transformed with respect to the Fourier Cosine transform

α = r transformed with respect to the zero order Hankel transform

J_0 = zero order Bessel function of the first kind

* = symbol for convolution

L^{-1} symbol for Inverse Laplace transform erf(x) = error function $\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy$

erfc(x) = complementary error function (1-erf(x))

$m(u,c) = \int_u^\infty \frac{e^{-\beta}}{\beta} \operatorname{erf}(c\sqrt{\beta})d\beta$, a tabulated function

$I(\alpha,\beta) = e^2 \int_\alpha^\infty \frac{e^{-\nu}d\nu}{\sqrt{\beta^2+\nu^2}}$, a tabulated function

$W(\alpha) = \int_\alpha^\infty \frac{e^{-y}}{y}dy$, well function, a tabulated function

$H(\beta) = \frac{\pi}{2}\left[H_0(\beta) - Y_0(\beta) \right]$

$H_0(\beta)$ = zero order Struve function

$Y_0(\beta)$ = zero order Bessel function of the second kind

Appendix I

The Fourier Cosine transform is

(1) $$Fc[v(x)] \equiv Vc(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty v(x) \cos \omega x \, dx$$

and the inversion formula is

(2) $$v(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty Vc(\omega) \cos \omega x \, d\omega$$

and

(3) $$Fc\left[\frac{\partial^2 V}{\partial x^2}\right] = - \omega^2 Vc(\omega) - \sqrt{\frac{2}{\pi}} \left[\frac{\partial V}{\partial x}\right]_{x = 0}$$

(4) Fourier Sine Inversion formula

$$v(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty vx(\omega) \sin \omega x \, d\omega$$

The Hankel transform of order zero is

(5) $$H_0[v(r)] \equiv V_0(\alpha) = \int_0^\infty rJ_0(\alpha r)v(r)dr$$

and the inversion formula is

(6) $$v(r) = \int_0^\infty \alpha J_0(\alpha r)V_0(\alpha)d\alpha$$

and

(7) $$H_0 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) \right] = - \alpha^2 F(\alpha) + \left[rJ_0(\alpha r) \frac{\partial f}{\partial r} + \alpha r J_1(\alpha r) f(r) \right]_{r=0}^{r=\infty}$$

The Laplace transform is

(8) $$L[f(t)] \equiv \bar{f}(p) = \int_0^\infty e^{-pt} f(t)\,dt$$

The convolution theorem is

(9) $$L^{-1} [f_1(p) f_2(p)] = \int_0^t F_1(t-\tau) F_2(\tau)\,d\tau$$

The addition theorem is

(10) $$L^{-1} [f(p-a)] = e^{at} F(t)$$

Appendix II

Laplace Transforms

	$F(t)$	$F(p)$
(1)	1	$1/p$
(2)	$\mathrm{erfc}\left(\frac{k}{2\sqrt{t}}\right)$ (k>0)	$\frac{1}{p}\, e^{-k\sqrt{p}}$
(3)	$\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{k^2}{4t}\right)$ (k>0)	$\frac{1}{\sqrt{p}}\, e^{-k\sqrt{p}}$
(4)	$-e^{ak} e^{a^2 t} \mathrm{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right) + \mathrm{erfc}\left(\frac{k}{2\sqrt{t}}\right)$ (k>0)	$\frac{a\, e^{-k\sqrt{p}}}{p(a+\sqrt{p})}$
(5)	$e^{ak} e^{a^2 t} \mathrm{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right)$ (k>0)	$\frac{e^{-k\sqrt{p}}}{\sqrt{p}(a+\sqrt{p})}$

Hankel Transforms

	$f(x)$	$\int_0^\infty f(x) J\nu(xy)(xy)^{1/2} dx$ (y > 0)
(6)	$x^{\nu+1/2} e^{-ax^2}$ (Rea > 0, Reν > -1)	$\frac{y^{\nu+1/2}}{(2a)^{\nu+1}} \exp\left(-\frac{y^2}{4a}\right)$
(7)	$x^{-3/2}(1-e^{-ax})$ (Rea > 0)	$y^{1/2} \mathrm{Sinh}^{-1}\left(\frac{a}{y}\right)$

(8)	e^{-ax}	$\frac{1}{\sqrt{a^2+y^2}}$

Fourier Sine Transforms

	$f(x)$	$g(y) = \int_0^\infty f(x)\sin(xy)\,dx \quad (y>0)$
(9)	$x^{-1}(x^2+a^2)^{-1}$ (Rea > 0)	$1/2\pi a^{-2}(1-e^{-ay})$

Fourier Cosine Transforms

	$f(x)$	$g(y) = \int_0^\infty f(x)\cos(xy)\,dx \quad (y>0)$
(10)	$(x^2 + a^2)^{-1}$ (Rea > 0)	$1/2\pi a^{-1} e^{-ay}$

$$s = \frac{Q}{4\pi K l}\left[\mathrm{Sinh}^{-1}\left(\frac{z-l}{r}\right) - \mathrm{Sinh}^{-1}\left(\frac{z+l}{r}\right) + 2\mathrm{Sinh}^{-1}\left(\frac{z}{r}\right) + 2e^{\alpha}\int_{\alpha}^{\infty}\frac{e^{-\nu}d\nu}{\sqrt{\beta^2+\nu^2}} - 2e^{\alpha}\int_{\alpha}^{\infty}\frac{e^{-\nu}d\nu}{\sqrt{\beta^2+\nu^2}}\right]$$

β	$\frac{l}{a}=1$, α=0	$\frac{l}{a}=1$, α=.2	$\frac{l}{a}=1$, α=0.4	$\frac{l}{a}=1$, α=0.6	$\frac{l}{a}=1$, α=0.8	$\frac{l}{a}=1$, α=1.0	$\frac{l}{a}=1$, α=1.5	$\frac{l}{a}=1$, α=2.0	$\frac{l}{a}=1$, α=3.0	$\frac{l}{a}=1$, α=4.0
.1	3.8190	4.3215	4.4766	4.4052	4.0081	2.7703	.8620	.4752	.2339	.1444
.2	2.6106	2.9901	3.1297	3.0719	2.7602	2.0938	.8410	.4708	.2293	.1446
.3	1.9600	2.2590	2.3819	2.3405	2.1107	1.6939	.8020	.4608	.2291	.1420
.4	1.5382	1.7796	1.8858	1.8617	1.6977	1.4196	.7567	.4491	.2244	.1551
.5	1.2398	1.4380	1.5308	1.5208	1.4052	1.2137	.7110	.4340	.2224	.1430
.6	1.0180	1.1828	1.2650	1.2666	1.1854	1.0488	.6678	.4194	.2205	.1416
.7	.8476	.9868	1.0601	1.0706	1.0139	.9148	.6243	.4023	.2173	.1421
.8	.7138	.8324	.8978	.9128	.8763	.8040	.5832	.3846	.2148	.1395
.9	.6070	.6970	.7683	.7869	.7636	.7093	.5360	.3668	.2093	.1396
1.0	.5200	.6085	.6622	.6836	.6701	.6284	.4752	.3514	.2045	.1386
1.1	.4490	.5290	.5750	.5780	.5911	.5597	.4398	.3344	.1977	.1365
1.2	.3902	.4578	.5026	.5262	.5242	.4998	.4095	.3280	.1919	.1359
1.3	.3416	.4010	.4375	.4635	.4675	.4492	.3776	.3000	.1839	.1352
1.4	.3000	.3530	.3896	.4134	.4180	.4062	.3466	.2814	.1777	.1339
1.5	.2650	.3121	.3456	.3688	.3755	.3576	.3216	.2645	.1741	.1285
1.6	.2350	.2775	.3083	.3305	.3467	.3306	.2844	.2508	.1678	.1289
1.7	.2092	.2472	.2757	.2972	.3057	.3064	.2729	.2262	.1659	.1259
1.8	.1870	.2216	.2451	.2671	.2779	.2775	.2513	.2172	.1571	.1223
1.9	.1678	.1986	.2226	.2430	.2526	.2511	.2326	.2059	.1518	.1181
2.0	.1510	.1510	.2014	.2210	.2303	.2352	.2211	.1908	.1446	.1112
2.5	.0932	.1109	.1260	.1417	.1474	.1523	.1525	.1454	.1103	.0984
3.0	.0606	.0730	.0834	.0957	.1029	.1095	.1162	.1094	.0991	.0800
3.5	.0408	.0495	.0568	.0677	.0732	.0786	.0892	.0888	.0783	.0647
4.0	.0286	.0348	.0402	.0492	.0538	.0558	.0666	.0718	.0636	.0541
4.5	.0200	.0246	.0288	.0369	.0404	.0458	.0505	.0545	.0536	.0383
5.0	.0140	.0177	.0210	.0281	.0300	.0334	.0384	.0445	.0429	.0432
5.5	.0102	.0131	.0156	.0216	.0244					
6.0		.0092								

Table 2

$$H(\beta) = \frac{\pi}{2} [H_0(\beta) - Y_0(\beta)]$$

β	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	$-\infty$	2.5095	1.8972	1.5651	1.3448	1.1845	1.6091	.96211	.88090	.81273
1	.75461	.70434	.66051	.62188	.58732	.55653	.52873	.50360	.48082	.45977
2	.44061	.42286	.40652	.39144	.37730	.36427	.35201	.34055	.32987	.31966
3	.31007	.30112	.29264	.28463	.27709	.26970	.26295	.25651	.25023	.24426
4	.23860	.23326	.22824	.22321	.21834	.21394	.20954	.20530	.20122	.19745
5	.19368	.19007	.18677	.18315	.18001	.17703	.17389	.17106	.16807	.16556
6	.16273	.16022	.15771	.15535	.15299	.15064	.14844	.14640	.14420	.14231
7	.14027	.13839	.13650	.13462	.13289	.13132	.12959	.12786	.12645	.12488
8	.12331	.12174	.12032	.11891	.11750	.11624	.11482	.11357	.11231	.11105
9	.10996	.10870	.10744	.10634	.10540	.10414	.10304	.10210	.10116	.10006
10	.98959	.98017	.97074	.96289	.95189	.94404	.93462	.92676	.91734	.91105
	(-1)	(-1)	(-1)	(-1)	(-1)	(-1)	(-1)	(-1)	(-1)	(-1)
11	.90163	.89377	.88592	.87807	.87021	.86236	.85608	.84979	.84194	.83408
12	.82780	.81995	.81366	.80738	.80110	.79481	.79010	.78225	.77597	.77125
13	.76497	.75869	.75240	.74769	.74141	.73670	.73198	.72570	.72099	.71470
14	.70999	.70685	.70214	.69585	.69117	.68643	.68172	.67701	.67229	.66915
15	.66287	.65973	.65501	.65030	.64716	.64402	.63774	.63459	.62988	.62517

Table 1

REFERENCES

Bateman, H., Tables of Integral Transforms, McGraw-Hill Book Co., New York, 1954.

Carslaw, H. S., and J. C. Jaeger, Operational Methods in Applied Mathematics, Dover Publications, Inc., New York, 1963.

Carslaw, H. S., and J. C. Jaeger, Conduction of Heat in Solids, Oxford University Press, 1959.

Churchill, Buel V., Operational Mathematics, McGraw-Hill Book Company, Inc., New York, 1958.

Churchill, Buel V., Fourier Series and Boundary Value Problems, McGraw-Hill Book Co., New York, 1963.

Dwight, Herbert Bristol, Mathematical Tables, Dover Publications, Inc., New York, 1961.

Ferris, J. B., D. B. Knowles, R. H. Brown, and R. W. Stallman, Theory of Aquifer Tests, U. S. Geol. Surv. Water-supply Paper 1536-E.

Hantush, M. S. and C. E. Jacob, Plane Potential Flow of Ground Water with Linear Leakage, Trans. Amer. Geophys. Union, 35, pp 917-936, 1954.

Hantush, M. S. and C. E. Jacob, Non-Steady Radial Flow in an Infinite Leaky Aquifer and Non-Steady Green's Functions for an Infinite Strip of Leaky Aquifer, Trans. Amer. Geophys. Union, 36, pp 95-112, 1955

Hantush, M. S., Analysis of Data From Pumping Tests in Leaky Aquifers, Trans. Amer. Geophys. Union, 37, pp 702-714, 1956.

Hantush, M. S., Non-Steady Flow to a Well Partially Penetrating an Infinite Leaky Aquifer, 1, pp 10-19, 1957.

Hantush, M. S., Non-Steady Flow to Flowing Wells in Leaky Aquifers, J.G.R., 64, pp 1043-1052, 1959.

Hantush, M. S., Modification of the Theory of Leaky Aquifers, J.G.R., 65, pp 3713-3725, 1960.

Hantush, M. S., Drainage Wells in Leaky Water-Table Aquifers, Proc. Am. Soc. Civil Engrs. 88, HY2, 123-137, 1962.

Hantush, M. S., Hydraulics of Wells in Advances in Hydroscience, Vol. 1, edited by Ven Te Chow, pp 281-432, Academic Press, New York, 1964.

Jacob, C. E., Radial Flow in a Leaky Artesian Aquifer, Trans. Amer. Geophys. Union, 27, pp 198-205, 1946.

McLochlan, N. W., Bessel Functions for Engineers, Oxford University Press, 1961.

Saad, Kamal Farid, Nonsteady Flow Toward Wells which Partially Penetrate Thick Artesian Aquifers, Master's thesis, New Mexico Tech, 1960.

Watson, G. N., Theory of Bessel Functions, The Macmillan Co., New York, 1944.

This thesis is accepted on behalf of the graduate faculty of the Institute by the following committee:

Date: 31 May 1966