AN ANALYTICAL INVESTIGATION OF SINUSOIDAL MICROBAROMETRIC OSCILLATIONS

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ABSTRACT

Previous studies strongly indicate that sinusoidal microbarometric oscillations, ranging in period from 5 to 10 minutes, with amplitudes less than 10^{-3} of the normal pressure field, are due to internal gravity waves, traveling either on a surface of atmospheric discontinuity (interface waves) or within a bounded duct (cellular waves). In order to determine which type of wave is the most probable mechanism responsible for producing the observed pressure perturbations, the present study considers wave motion in a temperature- and wind-stratified atmosphere. Mathematical analysis shows that the vertical wave number in any atmospheric layer may be real or imaginary, corresponding respectively to interface or cellular waves. Wave solutions can, in principle, be connected through multilayered regions, providing a complete mathematical description of the perturbed state in terms of the pressure perturbation measured at the ground. The analysis shows that special conditions are required in the layer in which the wind and wave velocities become equal. When certain simplifying assumptions hold true, this layer might be identified either as the interface (for real-valued vertical wave numbers) or as the upper boundary of a ground-based duct containing cellular waves (if the vertical wave number is imaginary).

Combining the results of the present analysis with the observed characteristics of microbarometric oscillations would indicate that to the extent the assumptions mentioned above are valid, a cellular wave motion is the probable cause of the observed sinusoidal variations in pressure.

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I. INTRODUCTION

1. Background of the Study

For over sixty years investigators have observed and recorded fluctuations in the atmospheric pressure field of the earth. The development of sensitive microbarographs has made routine the detection of pressure changes as small as 10^{-4} of the normal pressure field, occurring during time intervals of from 10^{-1} to 10^{5} seconds. An empirically derived power spectrum of atmospheric pressure extending over this period range has been published by Gossard (1960). Upon examination of the spectrum it is apparent that the high-frequency, generally non-periodic, "noise" is due to turbulence. The low-frequency portion of the pressure spectrum is charcterized principally, as would be expected, by the semidiurnal and diurnal lines.

In the present investigation attention is focused on the midfrequency range, and in particular on the narrow band of periods between 5 and 10 minutes. Atmospheric pressure variations in this period range frequently appear as remarkably uniform sinusoidal oscillations, in contrast with the random motion characteristic of the normal pressure record produced by a sensitive microbarograph. Figure 1 illustrates the relatively smooth and periodic character of these undulations.

Virtually all investigators who have studied pressure perturbations in the middle frequencies (corresponding to periods from one minute to one hour) have ascribed such perturbations to convective activity and internal gravity waves in the lower troposphere. Empirical studies conducted by the Research and Development Division of the New Mexico Institute of Mining and Technology indicate that convection and the influence of nearby weather systems are usually characterized by longer period (15 minutes to one hour) pressure fluctuations of rather irregular appearance. Shorter period sinusoidal pressure perturbations are undoubtedly related to tropospheric gravity waves. It appears that such pressure oscillations are but one manifestation of atmospheric wave motion. Wave clouds of various types, clear-air "turbulence" and certain periodic fluctuations in the propagation of radio and radar waves indicate that the atmosphere is frequently perturbed by wavelike motions.

2. Purpose of the Study

Previous studies strongly indicate that sinusoidal pressure variations recorded at the ground are due to tropospheric gravity waves. However, the type of wave motion and the physical mechanism underlying this motion have not been adequately described. Two types of gravity waves, either of which can produce the observed microbarometric oscillations, have

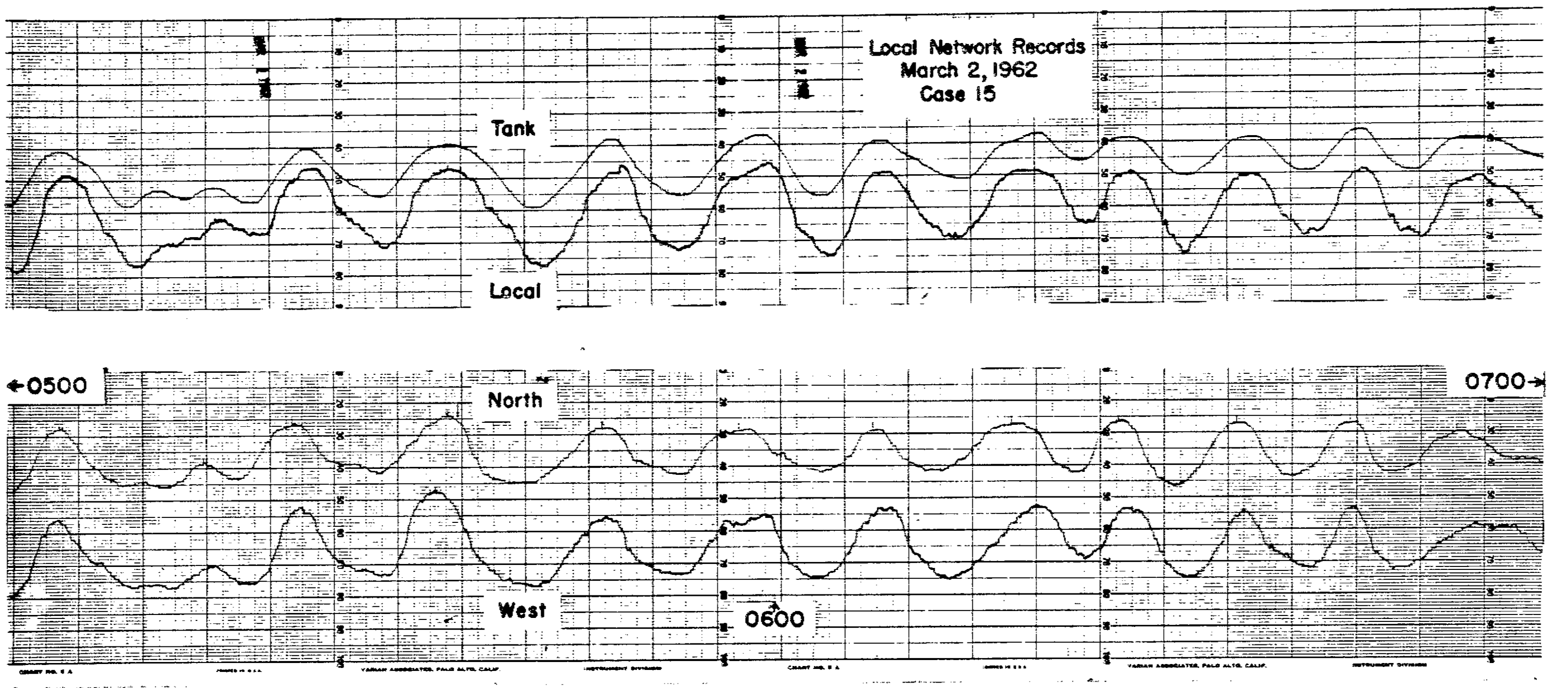

Figure 1. An illustration of a microbarometric oscillation recorded at four pressure stations. The average peak-to-peak amplitude is approximately 0.18 millibars.

been suggested:

(a) Interface waves, which travel along a tropospheric surface of discontinuity in temperature, temperature gradient, wind speed and/or direction, or some other atmospheric parameter;

(b) Cellular waves, which propagate vertically as well as horizontally. The term "cellular" means simply that the atmosphere appears divided into cells by the nodal surfaces of the wave motion.

The present study is directed specifically to the determination of the most probable mechanism responsible for producing sinusoidal pressure oscillations at the earth's surface in the period range of from 5 to 10 minutes.

3. Approach and Presentation

The equations governing motion in a temperature- and wind-stratified atmosphere will be stated and then simplified by perturbation methods to eliminate nonlinear terms. Expressions for the components of the velocity perturbation will be derived and combined into a second-order linear differential equation in the divergence of the velocity perturbation. The solution of this equation, in any atmospheric layer of known properties, indicates the type of wave motion (interface or cellular) occurring in that layer. Layers can be joined, through a connection matrix, so that the effects of the wave motion can be transmitted to the ground. The pressure perturbation measured at the earth's surface can in principle be combined with appropriate components of the connection matrix to give a complete description of the parameters of the perturbation at all heights.

The layer in which the wind and wave velocities become equal will be shown to possess special properties which indicate that this layer might function, if certain simplifying assumptions are valid, either as an interface or as the upper boundary of a ground-based duct containing cellular waves. The relationship of wind shear across the layer to the interface wave will be demonstrated.

The analysis presented here is unique in several respects. The undisturbed structure of the troposphere is defined to include the possibility of wind shear, a situation certainly observed in most cases but neglected in prior analytical studies. No initial assumption is made as to the type of gravity wave. Most authors have started their investigation by assuming the presence of either an interface or a cellular wave. The present study emphasizes the importance of the layer in which the wind and wave velocities become equal, for this layer appears to exert a fundamental influence on the type of wave motion producing microbarometric oscillations. However, the generality of the conclusions arrived at is limited by certain simplifying assumptions whose region of validity remains to be investigated.

II. CHARACTERISTICS OF THE MICROBAROMETRIC OSCILLATION

Sinusoidal microbarometric undulations have been the subject of numerous empirical studies, the results of which may be combined into a composite picture of the phenomenon. The essential features of the pressure oscillation are described below from an observational point of view. Readers not specifically interested in the detailed documentation following may turn to Table I (page 19) where the salient properties of the phenomenon are summarized.

1. Waveform

Probably the most immediately obvious characteristic of the microbarometric oscillations discussed in this study is their regular and uniform sinusoidal appearance. Early in this century, Shaw and Dines (1905) published a microbarogram showing "a series of very regular oscillations, ... obviously suggestive of atmospheric waves". The appearance of the phenomenon on any reasonably sensitive microbarograph is striking enough to ensure immediate identification. Johnson (1929) presents examples of sinusoidal waveforms with the comment, "oscillations with a period of two minutes (or even less) are extremely obvious and very easily detected in the records of any properly adjusted microbarograph".

Goldie (1925), Namekawa (1934-36), Haurwitz (1935), Gossard and Munk (1954), Fullerton (1964) and others have presented examples of clearly sinusoidal pressure oscillations, similar to Figure 1. Gossard (1962) has published some excellent microbarograms with the comment: "The records are very regular and essentially sinusoidal in character. It is a fair-weather phenomenon, rarely associated with storm fronts."

2. Periodicity

The typical case of sinusoidal microbarometric oscillations has an average period of between 5 and 10 minutes. It is necessary to refer to an "average" period since these variations in pressure rarely maintain a precisely constant period for more than two or three cycles. The changes in period are usually not drastic, although Johnson (1929) remarked:

> ...on some occasions a sustained oscillation is shown on a microbarogram and after perhaps half-an-hour it changes its period. The change may consist of either an increase or a decrease in the period. In a few cases the period increased to double its initial value, but these instances are exceptional rather than the rule. Normally, there seems to be no connection between the original and final periods.

Most investigators have studied a rather wide range of periods, usually between one minute and one hour. In almost all of these studies the predominately observed period has been found to be very close to the 5 to 10 minute period band. Brunt (1927) mentioned that "periods of this duration [between 6 and 10 minutes] are frequently observed on microbarograph traces..." Johnson (1929), who studied periods from 2 minutes to one hour, found a "well defined maximum frequency [of occurrence, at two different stations] for a period of ten minutes". Namekawa (1935), after a monumental ten year study, stated that "the period of maximum frequency at Kyoto is six minutes", with most periods between 3 and 12 minutes. Suzuki and Oomori (1937) reported typical periods of from 6 to 11 minutes and Köhne (1940) found most periods to lie in the range of from 5 to 15 minutes.

Flauraud, et al. (1954), as the result of a very elaborate empirical investigation of microbarometric oscillations, concluded: "The peak is in the period range the mean of which is ten minutes with a relative frequency of occurrence of 35.5 percent." Fullerton (1964), specifically restricting attention to the band of periods between 5 and 10 minutes, found the mean period to be approximately 7 minutes.

3. Phase Velocity

If a moving pressure oscillation can be identified across at least three stations of a nonlinear array, the apparent phase velocity of the disturbance can be calculated. The earliest such determination was made by Goldie (1925), who calculated a speed of 12 m/sec from the records of a 10 minute period oscillation recorded at five separate locations. Many investigators have quoted Goldie's phase speed figure as the typical value. However, Suzuki and Oomori (1937) found higher speeds, of the order of 20 to 50 m/sec, while Köhne (1940) reported a range of from 10 to 30 m/sec. Gossard and Munk (1954) studied a series of pressure oscillations with "phase velocities ... of the order of 10 m/sec." Fullerton (1964) found the mean of 25 cases to be 8 m/sec, with the disturbance moving generally in the direction of the prevailing lower tropospheric wind.

The consensus of these empirical findings is that the phase velocity of the phenomenon under discussion is only three to four percent of the Laplacian velocity of sound, making it clear that the disturbance producing microbarometric oscillations is not being propagated through the atmosphere with the characteristic speed of the medium (the speed of sound). Rather, typical phase velocities appear to be of the same order of magnitude as wind speeds in the lower troposphere.

4. Wavelength

Goldie (1925) calculated a wavelength of 7.2 km for the case mentioned in the preceding section. Gossard and Munk (1954) found wavelengths to be of the order of 4 to 10 km and Fullerton (1964) determined a wavelength range of from 0.8 to 7.3 km, the mean value being 3.3 km. Using the higher phase velocities they measured, Suzuki and Oomori (1937) found typical wavelengths of 13 to 25 km and Köhne (1940) obtained a wavelength range of from 10 to 40 km.

In any case, the wavelength of a sinusoidal microbarometric oscillation does not exceed 50 km and under almost all normal conditions it is considerably less.

5. Recorded Amplitude at the Earth's Surface

The range (that is, the peak-to-peak amplitude) of the microbarometric oscillation, as measured at the earth's surface, is usually of the order of 0.10 to 0.50 mb, although Goldie (1925) remarked that the "range of the variation may reach one millibar." Haurwitz (1935) discussed two cases of longer period waves, a 15 minute period oscillation having an amplitude of 0.22 mb and a 16 minute period fluctuation with an amplitude of 0.33 mb. Humphreys (1940) states that "small pressure changes, amplitude usually 0.1 to 0.3 mm [0.13 to 0.39 mb] and period of 5 to 10 minutes, and continuing for hours, or even days, together, are very common during cold weather." Köhne (1940) reports typical pressure amplitudes of 0.07 to 0.10 mm (0.09 to 0.13 mb).

The maximum range of the pressure excursion, one millibar (approximately 0.75 mm or 0.03 inches of mercury), is about 10^{-3} of the standard atmospheric pressure at sea level (1013.3 mb). Typical pressure amplitudes associated with microbarometric oscillations, as indicated above, are of the order of 10^{-4} of the standard atmospheric pressure. These amplitudes can be recorded easily by sensitive pressure sensors. Johnson (1929) observed that "in most cases oscillations are very well defined and nowhere near the limit of detectability."

6. Spatial Extent

If a pressure disturbance can be followed by phase identification from the records of one station to the records of another, a measure of the spatial extent of the disturbance can be obtained. Several investigators have examined this characteristic of microbarometric oscillations by establishing arrays of pressure sensors with varying station separations. Namekawa (1935), after analyzing the records from a three station network, stated: "Although the three observatories are not more than four kilometers apart ... they indicate comparatively different shapes of curve in some occasion." In discussing the 15 and 16 minute period oscillations previously referred to, Haurwitz (1935) mentioned that there was "no indication of these waves" recorded at another station only 18 km away. Flauraud, et al. (1954) reported: "The longer period (>15 minutes) oscillations usually could be identified across the ten mile [16 km] arrays; the shorter period (<15 minutes) oscillations generally could not." Fullerton (1964), comparing the size of arrays used by several investigators, concluded that the usual 5 to 10 minute period pressure waves probably maintain an identifiable waveform for an average distance of the order of only 5 to 7 km.

The loss of identity of the wave pattern over a distance of approximately one wavelength suggests that the phenomenon may be of quite limited horizontal extent normal to the direction of propagation. Furthermore, if the tropospheric wave motion occurs in a stratum located close to the ground (within, say, 1 to 3 km), as has been suggested by most investigators, the effects of local topography on wind and temperature structure may well affect the recording of the pressure fluctuation at the ground. The limited spatial extent of the phenomenon at ground level does not necessarily imply that the tropospheric wave motion is similarly restricted.

7. Surface Temperature Variations

Very little information has been published on the surface temperature conditions during the recording of sinusoidal oscillations in the pressure field. Matthews (1951) reported on a 15 minute period oscillation with initial amplitude of 0.8 mb. During the recording of this rather prominent pressure fluctuation the thermograph showed variations of less than 0.5°C. Matthews concluded that there was no marked change in the air mass. Fullerton (1964), using a thermister bridge temperature change sensor, capable of detecting changes as small as 0.03°C., found essentially no temperature variation during the recording of microbarometric waves.

8. Surface Wind Conditions

Theoretical considerations indicate that microbarometric oscillations may be accompanied by sinusoidal variations in the surface wind velocity. Investigations designed to verify this contention have not produced uniform results so that the simultaneous occurrence of pressure and wind oscillations appears to remain a possibility but not a certainty.

Over sixty years ago Shaw and Dines (1905) declared positively that "barometric oscillations are independent of the surface wind." Discussing a specific case they went on to say:

> It is true there is an obvious variation of the wind-force upon which transient gusts are superposed, and that there is a suggestion of periodicity about the more general variation, but the barometric variations persist after midnight when the wind has died away, and are still shown during the calm between 1 a.m. and 3 a.m.

Namekawa (1934) wrote:

> As for the correspondence of the waves of pressure and those of the wind, some English scientists say that at times waves of pressure and wind occur simultaneously. But not always so. The waves of pressure frequently occur without any corresponding oscillation of wind.

In a later paper Namekawa (1935) commented further:

> In our observatory at Kyoto, we have only a few examples of waves of pressure accompanied by oscillations of the wind. In many a case, however, we have waves of pressure without oscillations of the wind.

These observations were confirmed generally by Fullerton (1964) who found only one case of corresponding oscillations in wind speed among 36 cases of microbarometric fluctuations. In this one case there was no discernable periodic variation in the direction of the surface wind.

On the other hand, Haurwitz (1935) presents records clearly showing variations in the wind speed of amplitude 1.5 to 2 m/sec accompanying pressure oscillations of periods 15 and 16 minutes. The corresponding record of the surface wind direction contains the suggestion of periodic fluctuations. Flauraud, *et al.* (1954) report:

> In general, for waves with periods less than about ten minutes and amplitudes less than the maximum which can be recorded accurately, wind oscillations of from 4- to 10-mph [1.8 to 4.4 m/sec] amplitude per 0.1-mb amplitude of pressure oscillation were found almost continually. For longer periods, simultaneous variations of wind and pressure are not always evident, although there are indications of out-of-phase variations with wind amplitudes of about 1.5 mph [0.66 m/sec] per 0.1 mb.

It should be observed that the empirically determined values of the amplitude of the wind oscillation are generally very small. It seems probable that the standard cup anemometer might well lack the sensitivity necessary to detect the usually minor fluctuations in the wind speed.

9. Upper Level Winds

In all recent studies of microbarometric oscillations the investigator has stressed the necessity of knowing the upper air wind structure, at least in the lower portion of the troposphere. General knowledge of probable conditions aloft may be helpful, but any precise conclusions as to the exact nature of the wave motion are impossible without detailed information on wind velocities and wind shear as a function of height. This information is usually difficult to obtain at the time and location necessary.

A search of the literature reveals the paucity of concrete data. Gossard and Munk (1954) report that "the wave crests appear to be oriented normal to the wind shear between the upper and lower winds." Flauraud, et al. (1954) mention that

> ... throughout the 18 months during which records were collected, most of the largest amplitude pressure oscillations, excluding thunderstorm variations, were associated with large wind shear.

Further,

> It appeared that the greatest number of large, long period oscillations occurred during intervals in which the upper level winds were increasing, i.e., acceleration appeared to be a more important factor than speed.

The study performed by Flauraud and his associates indicated that there was good agreement between wind and pressure wave speeds and wind and pressure wave directions at levels higher than the 500-mb level (approximately 5.6 km above sea level). "The average wave speed was found to be equal to 0.9 average wind speed at the 200-mb level [approximately 12 km above sea level] ". Using a system of auto- and cross-correlation techniques, Flauraud, et al. (1954) determined that:

> ... only the 30-minute period peaks at the 700-mb level [approximately 3 km above sea level] were found to be significant enough to justify any assumption that the pressure variations of the 30-minute period could be more closely associated with wind speed variations at the 700-mb level than with winds at any other level.

It is the present author's opinion that, however sophisticated the statistical methods may have been, the questionable validity and accuracy of the input information and the rather large time differences between the various data taken argue against the direct application of these results to the case of microbarometric oscillations.

10. Duration

Duration refers to the total length of time a given pressure oscillation is recorded on the microbarogram. Johnson (1929) found microbarometric oscillations lasted "15 minutes to about 8 hours, although the great majority lasted for about one or two hours." Fullerton (1964) reported similar results, the average duration being slightly more than one hour, with the extreme value being 3.5 hours. Flauraud, et al. (1954) stated: "In calm weather, trains of short period (from 5 to 10 minutes) oscillations lasting 2 or 3 hours were sometimes observed during the 2200-0400 interval." Gossard (1962) mentions that "this type of tropospheric wave train is short, lasting less than 2 hours."

11. Diurnal Variation

One of the most extensively studied characteristics of microbarometric oscillations has been their distribution by hour of the day. There is a high degree of agreement between the reports published by various investigators working in a wide variety of geographical locations. Since the recording of this type of microbarometric oscillation is clearly a function of local time it appears virtually certain that the detection of the phenomenon at the ground is controlled principally by the existing atmospheric conditions in the vicinity of the recording station. As has been mentioned previously, this result does not imply that the tropospheric wave motion is similarly restricted to any localized area or time interval.

Johnson (1929) found that 50 percent of all microbarometric oscillations were recorded between 0000 and 0800, with 34 percent recorded in the interval 1600 to 2400. Only 16 percent of the variations were recorded during the day (0800-1600). Namekawa (1935) observed the maximum frequency of occurrence to be in the interval 0400-0600, the minimum number being recorded between 1000 and 1200. Namekawa reported: "Waves with periods 5-8 minutes or 9-12 minutes have the most regular distribution; waves of longer periods indicate less regularity in distribution." Stone (1935), summarizing Clayton's study of wavelike pressure oscillations, mentions "waves were considerably more frequent by night than by day, with maxima about 5 a.m. and 6 p.m. and a primary minimum between 2 and 5 p.m." Suzuki and Oomori (1937) found the maximum frequency of occurrence in the interval 0800 to 1000.

Flauraud, _et al_. (1954) stated: "The 0400-1000 interval shows a tendency toward low values [of occurrences] for the long period waves and high values for the 10-minute waves, whereas the 1600-2200 interval indicates a greater percentage of long period waves." Fullerton (1964) found the maximum recording activity of microbarometric oscillations of period from 5 to 10 minutes to be in the interval 0100 to 0700 (in fact, 75 percent of all occurrences were recorded during the period of low wind speeds between 0000 and 0800). Plotting the number of occurrences of pressure fluctuations as a function of local time, Fullerton found a rather gradual rise in the hourly frequency of occurrence between 2100 and 0100, with a very rapid decrease at 0800 (see Figure 2). This sharp cutoff in the

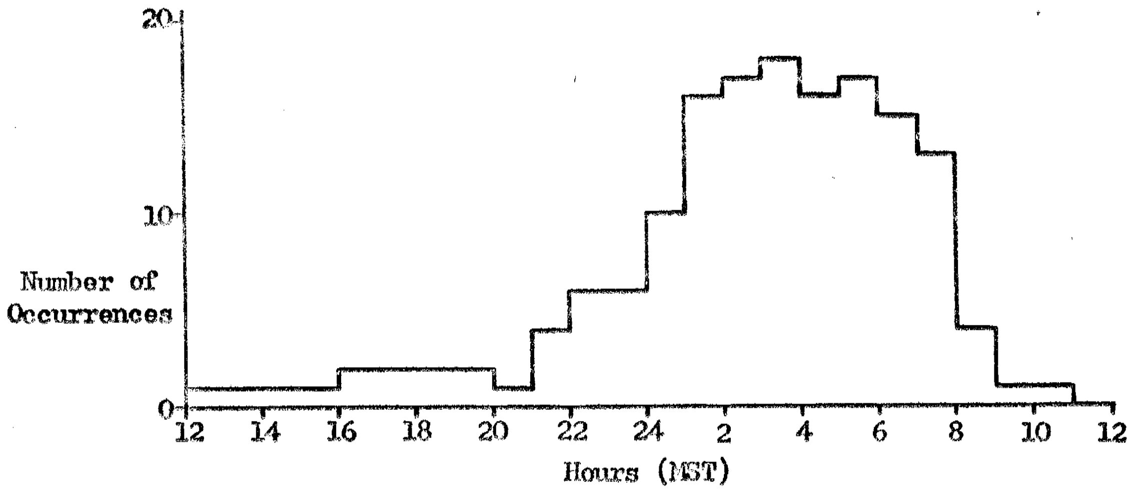

Figure 2. Diurnal variation in recorded microbarometric oscillations. (Fullerton, 1964)

morning appears to indicate that a fundamental change in atmospheric conditions (a steepening of the temperature lapse rate or the onset of convective activity) has occurred which serves to inhibit the recording of the wave phenomenon at the ground.

12. Seasonal Variation

Johnson (1929) found the following seasonal distribution of microbarometric oscillations:

Winter (December, January, February)	30 percent
Spring (March, April, May)	23 percent
Summer (June, July, August)	25 percent
Autumn (September, October, November)	22 percent.

Namekawa (1935) reported: "Occurrences are few in summer (July–September); for example the number in August is less than 3 percent of all." He mentions further: "Waves with period 5–8 minutes occur more frequently in the cold season." Namekawa (1935) concludes:

> The maximum frequency of short periodic wave(s) in the cold season or dawn suggests that these waves are the internal [that is, interface] waves, for the

> discontinuous surface between the surface and upper layers is formed very frequently at these times.

Suzuki and Oomori (1937) mention that microbarometric oscillations occur intensely during the cold season from September to April. Fullerton (1964) found the maximum frequency of occurrence during the interval February through April, with a secondary maximum in November.

The consensus of these findings is that periodic variations in pressure are more likely to be recorded during cold weather than during warm weather.

13. Synoptic Weather Patterns

Several comments in the preceding pages indicate that the pressure oscillations under discussion are essentially fair-weather phenomena. Goldie (1925) remarked: "Many cases of beautiful and regular waves occurred in anticyclonic [high pressure] conditions." Johnson (1929) distinguished between fair-weather microbarometric variations and those accompanying more disturbed synoptic conditions when he wrote:

> Disturbances associated with thunderstorms are quite different and are usually very heavily damped. In addition, there are certain quasi-periodic variations of pressure which occur on summer days and which appear to be connected with large scale convection. These again are totally distinct from the genuine oscillations which have been discussed in the present paper.

Suzuki and Oomori (1937) reported that fine regular waves usually appeared when calm weather had prevailed for some time.

For convenience, the several characteristics of the microbarometric oscillations to be considered in this study are summarized in Table I on the next page.

TABLE I. — Observed Characteristics

1. Waveform: Very regular and essentially sinusoidal in appearance.

2. Periodicity: Average period of between 5 and 10 minutes.

3. Phase Velocity: Approximately 10 m/sec, usually moving in the direction of the prevailing upper level winds.

4. Wavelength: Less than 25 km and more likely between 1 and 10 km.

5. Recorded Amplitude at the Earth's Surface: Peak-to-peak amplitude less than 1 mb and usually of the order of 0.10 to 0.50 mb.

6. Spatial Extent: Normally restricted to an average distance of 5-7 km.

7. Surface Temperature Variation: Essentially none.

8. Surface Wind Conditions: Simultaneous recording of periodic variations in surface wind speed and/or direction and microbarometric oscillations is a possibility but not a certainty.

9. Upper Level Winds: Wind velocity and wind shear in the lower troposphere are undoubtedly of major importance, but the limited empirical data available thus far do not permit specific evaluation of the extent of the influence.

10. Duration: A given pressure oscillation usually maintains an identifiable waveform for 1 or 2 hours.

11. Diurnal Variation: Most oscillations are recorded at night, very few in the day. The hourly frequency of occurrence drops sharply with the onset of morning heating (around 0800).

12. Seasonal Variation: Microbarometric oscillations are recorded more often in cold seasons than in warm seasons.

13. Synoptic Weather Patterns: Essentially a fair-weather phenomenon.

III. DOMINANT ATMOSPHERIC PARAMETERS

Certain properties of the troposphere have a direct influence on the phenomenon discussed in this paper. Any standard textbook on meteorology or atmospheric physics will provide a detailed description of these properties. However, for the convenience of the reader, those particular features of the troposphere which exert a dominant influence on the production and propagation of microbarometric oscillations will be briefly summarized in the following pages.

1. Stratification

Eckart (1960) has observed that

> The most striking phenomenon of the Earth's atmosphere is the diminution of its density with altitude. This stratification endows it with a stability that is completely lacking in a homogeneous fluid.

Two conclusions follow at once from this observation. First, the principles of classical hydrodynamics, which were developed for and are normally concerned with incompressible, homogeneous fluids, should be applied only with great care to the motions of a stratified, compressible atmosphere. Second, since the stratification introduced by a decrease of density with height imparts stability to the medium, the variation of density (or temperature) as a function of height may be expected to exert a dominant influence on the transmission and recording of microbarometric oscillations.

2. Atmospheric Stability

If a given layer of the atmosphere possesses attributes which tend to prevent or suppress vertical air motions, the layer is said to be stable or to exhibit stability. If conditions in the layer initiate vertical air motions or tend to accelerate those vertical motions present, the layer is considered to be unstable. Atmospheric stability is usually discussed only in terms of the thermal characteristics of the layer, since the environmental temperature structure plays the major role in the production and maintenance of vertical air motions. However, other factors, such as wind, wind shear and the presence of wave motion, must also be considered in the determination of atmospheric stability. The influence of each of these properties on stability will now be described briefly.

2-1. Thermal Stability

The dry adiabatic lapse rate, β_{ad}, is the rate of decrease of the temperature of dry air with height under conditions of adiabatic cooling:

$$\beta_{ad} = -\left(\frac{dT}{dz}\right)_{ad} = \frac{g}{c_p} = 10.0 \times 10^{-5} \text{ deg/cm} = 10 \text{ deg/km},$$

where T is the absolute temperature, z is the vertical coordinate measured positively upward from the earth's surface, g is the acceleration due to gravity (taken as a constant equal to 10 m/sec^2) and c_p is the specific heat at constant pressure of dry air. Observe that β_{ad}, as defined here, is a positive constant.

The rate of decrease of temperature with height, as actually measured in a given layer of the atmosphere, is called the environmental lapse rate, or simply the lapse rate β, so that

$$\beta = -\left(\frac{dT}{dz}\right).$$

The fundamental criterion for thermal stability is that β be less than β_{ad} and, in general, the greater the difference between β and β_{ad}, the greater the stability (if $\beta < \beta_{ad}$) or the greater the instability (if $\beta > \beta_{ad}$). An inversion layer, where the temperature increases with height, represents a particularly stable atmospheric condition.

2-2. Wind, Wind Shear and Wave Motion

The influence of the wind on atmospheric stability has not been established as precisely as the effect of the temperature gradient. However, some qualitative observations may be made. Kuettner (1952) has described the effect of the wind and wind shear on atmospheric stability as follows:

> If we consider a certain height and find that the lapse rate is stable at that elevation but that the wind speed is stronger above and smaller below this level, then this increase of wind with height tends to increase the stability of the atmosphere at the height considered. A decrease of wind with height has a destabilizing effect. However, in general, the thermal stability is much more effective than this wind shear term...
>
> More important is the curvature of the vertical wind profile, or, better, the rate of change of wind shear with height. If a "positive wind shear" (positive equals wind increasing with height; negative equals wind decreasing with height) is strongly increasing with height (or a negative decreasing), it can overcompensate the thermal stability and produce instability. As a consequence, convection can develop even in a thermally stable atmosphere. However, it might be preferable to call that "turbulence". On the other hand, a decreasing positive wind shear (or an increasing negative) might have a strong stabilizing effect.
>
> There is still another effect of the wind on stability. In comparing two levels with different wind speeds but with the same lapse rate, it will be found that the higher the wind speed, the lower the stability, regardless of the wind shear terms. Because of this fact and the large lapse rate in the upper troposphere, we find generally a minimum of stability just below the tropopause.

In addition to the influences of temperature gradient, wind and wind shear on atmospheric stability, Kuettner (1952) reports that the presence of wave motion also affects stability:

We assume now a wave motion in the atmosphere which propagates horizontally. It turns out that the stability in a certain level is not the same for different wavelengths, the general tendency being the smaller the wavelength, the smaller the stability.

3. The Väisälä-Brunt Frequency

It has been noted previously (section 2-1) that the degree of thermal stability of a given layer of the atmosphere is determined by the difference between the dry adiabatic lapse rate (β_{ad}) and the environmental lapse rate (β) in that layer. This criterion may be expressed in mathematical terms by considering the oscillations of a small parcel of the atmosphere when it is displaced from its equilibrium position and permitted to move freely. The derivation of the resulting frequency was first carried out by Väisälä in 1925 and, independently, by Brunt (1927).

In the derivation given by Eckart (1960), the bucyancy force experienced by the parcel is equated to the inertial reaction of the parcel with the resulting equation of motion:

$$\frac{d^2z}{dt^2} + N^2z = 0,$$

where N is the Väisälä-Brunt (or bucyancy) frequency.

The Väisälä-Brunt frequency may be defined in a variety of equivalent forms. For example,

$$N^2 = \frac{g}{T}(\beta_{ad} - \beta) = -g\left(\frac{1}{\rho}\frac{\partial\rho}{\partial z} + \frac{g}{c^2}\right), \tag{1}$$

where ρ is the density and c is the velocity of sound at height z. For N to be real it is necessary that $\beta < \beta_{ad}$. Under these conditions the atmospheric layer under discussion is stable and a small parcel of air will execute oscillations of frequency N (hence, of period $\frac{2\pi}{N}$ seconds). If N is imaginary, the air parcel will be unstable and deviate permanently from its equilibrium position.

Equation (1) may be written in the form:

$$N^2T = g(\beta_{ad} - \beta),$$

where the right-hand side depends only on the observed temperature gradient and so can be readily evaluated. Table II provides the value of N^2T for a number of typical lapse rates, and the resulting value of N^2 for a series of selected temperatures. Figure 3 presents graphically the variation of N^2 as a function of lapse rate for this same series of temperatures.

TABLE II. - Values of N^2 for Selected β and T

β (deg/km)	N^2T (deg/sec^2)	N^2 (10^{-4} sec^{-2})		
		T = 280°	T = 290°	T = 300°
-10	0.20	7.14	6.90	6.67
- 5	0.15	5.36	5.17	5.00
- 1	0.11	3.93	3.79	3.67
0	0.10	3.57	3.45	3.33
1	0.09	3.21	3.10	3.00
2	0.08	2.86	2.76	2.67
3	0.07	2.50	2.41	2.33
4	0.06	2.14	2.07	2.00
5	0.05	1.79	1.72	1.67
6	0.04	1.43	1.38	1.33
7	0.03	1.07	1.03	1.00
8	0.02	0.714	0.690	0.667
9	0.01	0.357	0.345	0.333
10	0.00	0.00	0.00	0.00

In this table, $\beta_{ad} \equiv 10$ deg/km and $g \equiv 10$ m/sec^2.

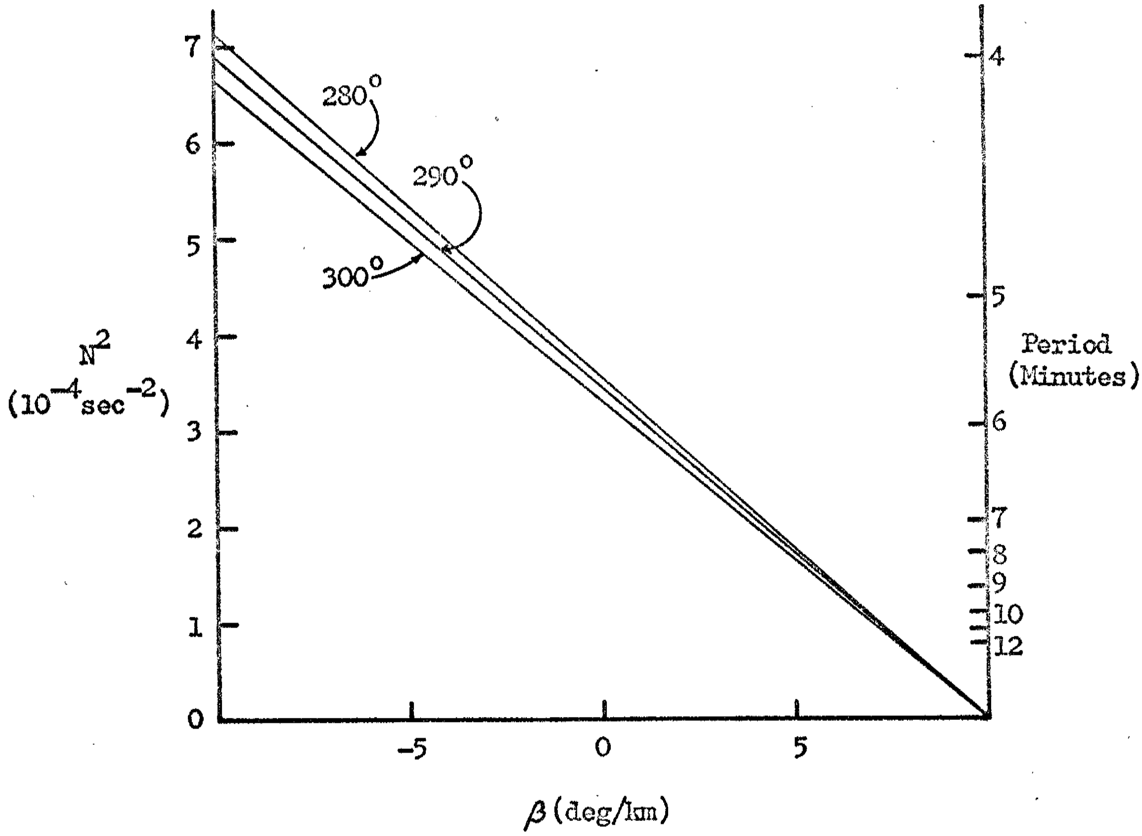

Figure 3. Väisälä-Brunt frequencies and periods associated with various values of β.

Figure 3 shows that the variation of N^2 with temperature is small. For example, at the most commonly observed lapse rate (6.67 deg/km), N^2 varies only 7×10^{-6} sec^{-2} over a range of 20°. Gossard (1962) mentions that, since N depends principally on the height gradient of density rather than on the density itself, some authors have considered N to be constant over the whole depth of the troposphere. Eckart (1960) suggests that N may summarize the small-scale dynamics of adiabatic expansion.

Since this study is primarily concerned with microbarometric oscillations, it is of interest to examine the Väisälä-Brunt periods. The scale on the right-hand side of Figure 3 gives the period in minutes associated with the frequency N. It is immediately obvious, as Brunt (1927) noted, that "the periods of oscillation which correspond to the most frequently observed lapse rates lie between 6 and 10 minutes". Since the most commonly recorded periods of microbarometric oscillations lie in the range of from 5 to 10 minutes, it seems apparent that these pressure variations are closely related to existing atmospheric lapse rates.

4. Discontinuity Surfaces and Ducts

Two air masses with different physical properties may exist in the same general region of the atmosphere. The transitional zone separating these air masses is known as a discontinuity surface or front. In many analytical treatments the front is taken as a mathematical discontinuity in density (or, equivalently, in temperature). Haurwitz (1964) has commented that such an idealization will not produce too serious an error provided that "the wavelength is sufficiently much larger than the size of the transition zone which is replaced by the discontinuity in the mathematical model."

In addition to density-temperature discontinuities, any marked changes in the motion of air masses with height may also introduce a surface of separation. Such vertical wind shears contribute to the motion of fronts so that any moving front will probably exhibit discontinuities in both density and wind velocity.

Since the boundary between two air masses is frequently a temperature discontinuity it is probable (although not essential) that the air masses will have different temperature gradients. Near the surface of the earth and with clear skies generally stable conditions exist at night, with unstable conditions more likely during the day. Low level temperature inversions are particularly prevalent at night (nocturnal inversions) and, at continental stations, during the colder seasons (during the warmer seasons at maritime stations). Byers (1944) observes that "cold fronts can occur aloft above a perfectly horizontal and therefore non-frontal inversion," and that such conditions appear with greatest frequency in the winter.

The presence of a marked temperature inversion at the ground effectively traps the surface air beneath a "lid". Horizontal air movement can occur within this surface zone, but very little or no vertical motion is possible. Thus, a surface duct or wave guide is formed, bounded below by the ground and above by the temperature inversion cover. Stable layers aloft, confined above and below by unstable regions, may also serve as ducts in which wave motions are effectively trapped. Martyn (1950) suggests that adequate duct boundaries are "more readily [that is, effectively] formed by a small increase in lapse rate than by a small increase in temperature..."

5. Weather Systems and the Jet Stream

Many types of weather systems are accompanied by periodic fluctuations of pressure. In general, however, wavelike variations of the pressure field directly associated with disturbed synoptic patterns are of longer period and of much more irregular appearance than the microbarometric oscillations discussed in this study. Normally, it is possible to distinguish between these two classes of periodic motion and to concentrate attention on the phenomenon of primary interest. In the present study only the fair-weather pressure variations described previously will be considered.

However, it should be realized that tropospheric wave motions can be and frequently are initiated by disturbed weather conditions, such as thunderstorms and the movements of high and low pressure systems. In these cases the tropospheric waves may have traveled a considerable distance from the region of generation to the location of detection. The point stressed here is that in the vicinity of the microbarograph site weather conditions may be generally described as quiescent.

The influence of jet streams on the production or detection of microbarometric oscillations remains questionable. A jet stream is characterized principally by high wind speeds (at least 30 m/sec) and strong wind shear. It is known that certain traveling waves with average periods between 20 and 30 minutes (Kuettner, 1952) are closely associated with jet stream conditions in the high troposphere. These "jet stream waves" may occasionally be recorded on sensitive microbarographs but it seems unlikely that sinusoidal pressure variations of periods between 5 and 10 minutes are routinely associated with major jet stream systems.

On the other hand, there are transient, low altitude local jet streams which may play a role in the production of atmospheric wave motion. Barad (1961) calls these winds the "low-level nocturnal jet", since they usually build in the afternoon, reach their maximum speed in the middle of the night, and decay in early morning. Such jets usually develop at heights of only a few hundred meters above flat terrain so that they may well have some association with microbarometric oscillations. However, this possibility has yet to be established.

IV. A CONSIDERATION OF POSSIBLE MECHANISMS

In the following pages brief consideration is given to a number of possible mechanisms which may contribute to pressure variations recorded at the earth's surface as microbarometric oscillations. These general comments are meant to suggest possibilities rather than as comprehensive reviews of the several theories proposed by various investigators.

1. Vertical Oscillations of a Bubble of Air

Probably the first and most obvious physical mechanism which might conceivably produce sinusoidal variations in the microbarograph record would be the simple vertical oscillations of a bubble of air. This possibility follows immediately from the close correspondence between the observed periods of the pressure fluctuation and the Väisälä-Brunt periods.

If the pressure sensor is responding to a simple vertical oscillation of an air bubble, the moving bubble apparently does not come into contact with the ground in the vicinity of the microbarograph station. This conclusion is derived from the observed constancy of the air temperature, which indicates that the macroscopic density gradient is not varying at the ground.

Observations also show that the disturbance causing the pressure fluctuations is apparently moving, suggesting that the oscillating bubble is not executing purely vertical motions unless it is located in the stratum where the wind and wave velocities are equal. In this case the bubble would be at rest in the horizontal direction, with respect to the moving air, but free to oscillate vertically if the stratum possesses the necessary stability ($N^2 > 0$). It will be shown later that when $N^2 > 0$ at the level where the wind and wave velocities are equal, the vertical component of the velocity perturbation is zero at this level. Thus, vertical motions are not present at this level, suggesting that the pressure sensor apparently is not responding to the simple vertical oscillations of a bubble of air.

2. Mountain Waves

Numerous studies indicate that air flowing down the lee slope of a mountain may be expected to execute one or more vertical oscillations (lee waves) before resuming horizontal flow. Arguments have been advanced both for and against the identification of lee waves as a possible source of microbarometric oscillations. The reasons for opposing this identification will be presented first.

The fundamental argument given against the association of microbarometric oscillations with lee waves is based on definition: a lee wave is a standing or stationary wave which remains fixed in space with respect to that barrier in the upper level air flow which originally initiates it. Since it has been established by a number of independent investigations that microbarometric oscillations are probably related to progressive wave systems, the possibility of stationary lee waves being the direct source of such pressure fluctuations seems very doubtful. Perhaps an even more telling argument is the simple observation that microbarometric oscillations are frequently recorded at stations far removed from any orographic feature capable of initiating lee waves. This fact alone assures the insufficiency of lee wave theory as the sole explanation of sinusoidal pressure variations. Finally, as far as the present writer knows, there is no published report of any pressure variation measured at ground level which was positively identified as being due to lee waves. On occasion, clouds of the lee wave type have been observed above the microbarograph array of the

New Mexico Institute of Mining and Technology, with no indication of perturbations on the pressure record.

In spite of these cogent reasons for not associating lee waves with microbarometric oscillations it is the opinion of the author that at least some of the pressure variations recorded at stations located within the sphere of influence of major terrain features are due, directly or indirectly, to the presence of lee waves. It is certainly possible for a standing wave near the earth's surface to produce pressure fluctuations at a fixed ground station, provided that the station is not located directly below a nodal point. It is also conceivable that a standing wave pattern, once established, could become separated from the causal barrier and move "downstream" with the velocity of the upper level wind. Colson and Lindsay (1959) suggested such an interpretation for a series of barograph "decreases in pressure" observed in conjunction with the appearance of an "unusual wave cloud" over Washington, D.C.

Some of the characteristic lee wave scale sizes are very similar to those associated with microbarometric oscillations. For example, typical lee waves have wavelengths of between 1 and 10 km. On the basis of theoretical calculations Scorer (1953) concludes that lee waves may cause pressure perturbations of between 1 and 2 mb, while Roper (1952) suggests the pressure difference between the crest and trough of a lee wave system to be of the order of one millibar. In addition, there are secondary effects associated with some lee wave systems, such as eddies and rotors, which may produce pressure perturbations similar to those discussed in this report. In any event, since the disturbing influence of a hill or mountain is difficult to establish precisely, the possibility of an orographic origin of microbarometric oscillations should be considered.

3. Internal Gravity Waves

Virtually all investigators of microbarometric oscillations have agreed that the phenomenon recorded at ground level is fundamentally the consequence of a tropospheric gravity wave. Furthermore, consideration of the observed phase velocities confirms that the disturbance is of the internal wave type, rather than an external wave motion which exhibits acoustic modes of propagation. There also appears to be nearly unanimous agreement that the tropospheric wave motion derives its motive force from the kinetic energy of the wind, the restoring force being furnished by buoyancy. Such wave systems may be called shearing-gravity waves.

There are apparently two types of internal waves which seem to possess all of these properties, each of which may produce sinusoidal variations of pressure at ground level. These two types are discussed briefly below.

3-1. Interface Waves

In analogy with wind-generated waves on the surface of the ocean, Helmholtz (1889) suggested that atmospheric wave motion might be expected on discontinuity surfaces in the lower troposphere. Although Helmholtz had only the visual evidence provided by occasional wavelike cloud formations, the subsequent development of sensitive pressure sensors soon confirmed the presence of apparently well-ordered atmospheric wave motion. As knowledge of the troposphere grew it was established that discontinuity surfaces, capable of supporting such wave motion, were a fairly common feature. For these reasons, gravity waves propagating on a surface of discontinuity in the troposphere, which may be called interface waves, have become the most frequently given explanation of the disturbance which manifests itself at ground level as a microbarometric oscillation.

Mathematical analysis shows that the amplitude of an interface wave decreases rapidly with distance from the interface. Therefore, the discontinuity surface must be located fairly low in the troposphere, perhaps only one or two kilometers above the ground. Whenever a surface of discontinuity is known or suspected to be present, the possibility of interface waves must be considered. Knowledge of the temperature and wind structure of the lowest layers of the atmosphere will permit the determination of whether an interface wave of given amplitude can be detected at surface pressure monitoring stations.

3-2. Cellular Waves

If the atmospheric wave motion is not concentrated at a discontinuity surface the wave may be expected to propagate vertically as well as horizontally. This type of wave motion was first suggested by Martyn (1950). Since the atmosphere appears divided into cells by the nodal surfaces of the wave motion, Martyn referred to this possible source of microbarometric oscillations as a cellular wave.

It is probable that cellular waves travel horizontally in a duct or wave guide. In order to affect a surface microbarograph the lower boundary of the duct must be the ground. To compensate for the frictional effects at the ground, which might lead to a more rapid dissipation of the wave energy than is observed, a constant input of energy is required. The probable source of this energy is the increase of wind speed and wind shear with elevation. The upper boundary of the duct is formed by appropriate alterations of the dominant atmospheric parameters. Adequate bounding must exist at this upper surface to reduce excess leakage of wave energy into the higher altitudes. The presence of an upper boundary does not necessarily imply the existance of a discontinuity surface. It appears that rather minor changes in the atmospheric parameters, such as a slight increase in the temperature lapse rate or wind speed, may be sufficient to provide the necessary vertical bounding.

Both interface and cellular waves may produce sinusoidal pressure perturbations at the earth's surface. Therefore, a knowledge of conditions in the lower troposphere is necessary to distinguish which type of wave actually causes a given microbarometric oscillation.

V. ASSUMPTIONS AND DEFINITIONS

The preliminary assumptions and definitions required for an analytical treatment of microbarometric oscillations will now be explicitly stated. Additional restrictions and specialized terms will be introduced as required during the theoretical development.

1. Experimental Techniques

No consideration will be given to experimental apparatus or recording techniques. The similarity of microbarograms, produced by a variety of pressure monitoring instruments located at diverse sites throughout the world, assures that a particular phenomenon has been and will continue to be easily identified. It is assumed that whatever instrumentation may be employed, the resulting record presents a relatively undistorted and essentially accurate representation of the pressure perturbation occurring at that location.

2. Curvature and Rotation of the Earth

The observed order of magnitude of the wavelength and phase velocity of the tropospheric disturbance, as well as the amplitude of the pressure variation, indicates that the curvature and rotation of the earth may be disregarded. The approximation of a perfectly flat, stationary earth appears to be justified for the treatment of small wavelength phenomena. Scorer (1949) suggests that a small wavelength may be considered as any wavelength less than 50 km. By avoiding the complications of a rotating earth it is possible to omit the Coriolis force terms from the equation of motion.

3. The Undisturbed Atmosphere

While it is recognized that the atmosphere is a variable mixture of several gaseous components, the assumption will be made that it may be treated as a single dry gas of uniform composition and molecular weight. This assumption is probably justified if the region of the atmosphere under study does not extend to great heights. Glasstone (1965) suggests that the composition and average molecular weight of the atmosphere are essentially constant to an altitude of approximately 100 km. Since the phenomenon discussed in this report is almost certainly limited to the lower levels of the troposphere the assumption of an atmosphere of uniform composition appears to be warranted.

The air is considered to be dry so that the complications arising from variable water vapor content and the heat exchange processes associated with phase changes can be disregarded. This is a rather drastic assumption since many atmospheric processes are largely controlled by such energy adjustments. However, microbarometric oscillations appear to be related primarily to mechanical processes so that the supposition of inert air should not produce serious error.

In view of the above assumptions, air may be considered an ideal gas and the ideal gas law used as the equation of state. The ideal gas law may be written in the convenient form,

$$p = \rho R_m T, \tag{2}$$

where p is the pressure and R_m is the specific gas constant, that is, the universal gas constant divided by the molecular mass. If dry air is considered to be a mixture of "dry gases" forming one "gas" of molecular mass 28.9 grams/mole, $R_m = 2.88 \times 10^6$ ergs/gram-deg.

It is assumed that the undisturbed pressure structure is given by the hydrostatic equation,

$$\frac{\partial P}{\partial z} = - \rho_z g, \qquad (3)$$

where the undisturbed pressure P and the undisturbed density ρ_z are functions of the vertical coordinate z only.

It is assumed further that the undisturbed wind structure consists of a uniform, horizontal (but not independent of z) air flow defined by

$$\vec{V} = V_x \vec{i} + V_y \vec{j} + 0\,\vec{k}\ , \qquad (4)$$

where $\vec{i}$, $\vec{j}$, $\vec{k}$, are unit vectors in the direction of the coordinate axes x, y, z. The horizontal components V_x and V_y are functions of z only so that the above definition implies that the wind velocity $\vec{V}$ at any given elevation is constant, but that vertical wind shear is permitted. Considering the atmosphere as a series of horizontal layers, such a definition seems physically reasonable. Measurements indicate that typical undisturbed vertical velocities are at least two orders of magnitude less than horizontal velocities so that it appears valid to take the coefficient of the $\vec{k}$ term in equation (4) as zero.

The Laplacian velocity of sound, c, appropriate under adiabatic conditions, may be formulated in a number of mathematically equivalent definitions:

$$c^2 = \frac{\gamma P}{\rho_z} = \gamma R_m T = \gamma g H, \qquad (5)$$

where γ is the ratio of the specific heats at constant pressure and constant volume of dry air ($\gamma = 1.4$, a constant). Equation (5) indicates that c, a function of z only, may be expressed in terms of the absolute temperature at height z, T, or in terms of H, the so-called "height of the homogeneous atmosphere". If the atmosphere is assumed to be isothermal, the scale height of the isothermal (homogeneous) atmosphere, defined by

$$H = R_m \frac{T}{g} = \frac{P}{g \rho_t} , \quad (6)$$

is also constant. H is 8.0 km if T is 273°K, 8.6 km if T is 293°K. H is usually used as a convenient symbol, rather than as a basic parameter specifying a physically measurable portion of the real atmosphere.

4. The Disturbed Atmosphere

It is assumed that the quiescent atmosphere, as described in the preceding section, is perturbed by a tropospheric air flow defined by

$$\vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k} , \quad (7)$$

where v_x, v_y, v_z, are the components of the perturbation velocity $\vec{v}$. In accordance with observations, $\vec{v}$ will be assumed to vary sinusoidally in a manner which will be precisely specified later.

Some of the early studies of microbarometric oscillations included the assumption that the atmosphere was incompressible. Such a supposition greatly simplified the mathematical treatment, permitting the direct application of the principles of hydrodynamics. In particular, the equation of continuity was reduced to an almost trivial form.

In the theoretical considerations following, the atmosphere will be taken as compressible, as it is in reality. The complete expression of the equation of continuity will be used. It will be assumed further that the pressure variations involved take place under adiabatic conditions, so that the relationship between pressure and density changes is given by the adiabatic law to be derived.

5. Friction and Turbulence

The frictional terms in the equation of motion will be disregarded. This decision is based partly on a desire to confine the mathematical treatment to observable quantities, but mainly because the majority of microbarograms available do not exhibit damping of the pressure amplitude which might be expected if friction plays a dominant role. Undoubtedly, the conversion of well-ordered wave energy into small-scale turbulence takes place. However, this refinement will not be considered.

The relationship between turbulence and the generation and recording of microbarometric oscillations has not been established. A close association may well exist and future investigations should explore this possibility. Perhaps the present study, including as it does the possibility of wind shear, may be considered a first step in this direction.

6. Perturbation Methods

It will be assumed that perturbation theory can be used to linearize the basic hydrodynamic equations. This implies that only very small departures from the initial equilibrium state will be considered. It would seem that the symmetry and small amplitude character of microbarometric oscillations would make the application of perturbation methods to this case particularly appropriate.

Perturbation equations are derived basically under the assumption that the total atmospheric motion may be regarded as the sum of an undisturbed motion and a superimposed perturbation. It follows from this supposition that not only the total motion, but also the undisturbed motion, must satisfy all the equations and conditions appropriate to the problem under discussion. By subtracting the effect of the undisturbed motion from the total motion the influence of the perturbation becomes evident. It is assumed further that the resulting deviation from the undisturbed state is so small that products of perturbation quantities and their derivatives may be disregarded in comparison to those terms which are of first-order in the perturbation quantities.

Perturbation methods have been used with marked success in some fields (for example, astronomy and quantum mechanics) and with somewhat less spectacular results in others. It appears to be very difficult to predict in advance how closely a solution, based on perturbation theory, will represent reality. Eckart (1960) has suggested that at least some of the failures of perturbation theory may be due to the fact that it has "been used with inadequate care". An effort will be made in the following treatment to avoid this criticism.

VI. THE ANALYTICAL TREATMENT

1. Basic Equations

1-1. Equation of Motion

The general hydrodynamic equation of motion, applicable to any atmospheric motion $\vec{V}_o$, occurring at the level where the pressure is P_o and the density ρ_o, may be expressed (for example, by Craig, 1960) as

$$\frac{\partial \vec{V}_o}{\partial t} + \vec{V}_o \cdot \nabla \vec{V}_o = -2\vec{\Omega} \times \vec{V}_o - \frac{1}{\rho_o} \nabla P_o - \nabla \Phi + \vec{F}_m , \qquad (8)$$

where ∇ is the vector differential operator

$$\nabla \equiv \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} ,$$

and $2\vec{\Omega} \times \vec{V}_o$ is the Coriolis force, which will be disregarded. If Φ is the total gravitational potential, $\nabla \Phi$ is the gravitational force per unit mass, so that

$$\nabla \Phi \equiv 0\,\vec{i} + 0\,\vec{j} + g\,\vec{k} \equiv (0,0,g) ,$$

with g considered constant. $\vec{F}_m$ represents the total frictional force per unit mass, which will also be disregarded. Then, equation (8) becomes

$$\frac{\partial \vec{V}_o}{\partial t} + \vec{V}_o \cdot \nabla \vec{V}_o = -\frac{1}{\rho_o} \nabla P_o - (0,0,g) . \qquad (9)$$

It is now assumed that the atmospheric conditions actually observed, characterized by the quantities $\vec{V}_o$, P_o and ρ_o, are determined by the addition of a small perturbation to the undisturbed atmosphere. Thus,

$$\vec{V}_o = \vec{V} + \vec{v} = (V_x + v_x)\,\vec{i} + (V_y + v_y)\,\vec{j} + v_z\,\vec{k} ,$$
$$P_o = P + p \quad \text{and} \quad \rho_o = \rho_s + \rho , \qquad (10)$$

where $\vec{v}$, p and ρ are the parameters of the perturbed state. Substituting the relationships defined in (10) into the simplified equation of motion

(9), and recalling that $\vec{V}$ is not a function of time, gives

$$\frac{\partial \vec{v}}{\partial t} + (\vec{V} + \vec{v}) \cdot \nabla (\vec{V} + \vec{v}) = - \frac{1}{\rho_s + \rho} \nabla (P + p) - (0,0,g) \ . \qquad (11)$$

This equation may be linearized by the methods of perturbation theory (the details are given in Appendix I) and written in component form as three equations of motion:

$$\frac{\partial v_x}{\partial t} + V_x \frac{\partial v_x}{\partial x} + V_y \frac{\partial v_x}{\partial y} = - \frac{1}{\rho_s} \frac{\partial p}{\partial x} - \frac{\partial V_x}{\partial z} v_z \ ,$$

$$\frac{\partial v_y}{\partial t} + V_x \frac{\partial v_y}{\partial x} + V_y \frac{\partial v_y}{\partial y} = - \frac{1}{\rho_s} \frac{\partial p}{\partial y} - \frac{\partial V_y}{\partial z} v_z \ , \qquad (12)$$

$$\frac{\partial v_z}{\partial t} + V_x \frac{\partial v_z}{\partial x} + V_y \frac{\partial v_z}{\partial y} = - \frac{1}{\rho_s} \frac{\partial p}{\partial z} - \frac{g}{\rho_s} \rho .$$

1-2. Equation of Continuity

The equation of continuity, a statement of the law of conservation of mass for the compressible atmosphere, may be expressed (for example, by Craig, 1960) as

$$\frac{\partial \rho_o}{\partial t} + \nabla \cdot \rho_o \vec{V}_o = 0 \ . \qquad (13)$$

Substituting the relationships defined by equations (10) and observing that ρ_s is not a function of time, equation (13) becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_s + \rho)(\vec{V} + \vec{v}) = 0 \ . \qquad (14)$$

Application of perturbation methods (see Appendix II) reduces equation (14) to the linearized form of the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + v_z \frac{\partial \rho_s}{\partial z} + \rho_s \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0 \ . \qquad (15)$$

1-3. Adiabatic Relationship

The relationship between the so-called individual rate of change $\frac{d}{dt}$, the local rate of change $\frac{\partial}{\partial t}$, and the advective rate of change $\vec{V}_o \cdot \nabla$ is given (for example, by Holmboe, et al., 1945) as

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{V}_o \cdot \nabla \ .$$

Writing this operator identity for the total pressure P_o and the total density ρ_o gives

$$\frac{dP_o}{dt} = \frac{\partial P_o}{\partial t} + \vec{V}_o \cdot \nabla P_o \quad \text{and} \quad \frac{d\rho_o}{dt} = \frac{\partial \rho_o}{\partial t} + \vec{V}_o \cdot \nabla \rho_o \ . \tag{16}$$

Substituting the relationships from equation (10) into equations (16) and observing that P and ρ_s are not functions of time, yields

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + (\vec{V} + \vec{v}) \cdot \nabla (P + p)$$

and

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\vec{V} + \vec{v}) \cdot \nabla (\rho_s + \rho) \ .$$

Each of these expressions may be expanded and the appropriate undisturbed terms subtracted. If the terms $\vec{v} \cdot \nabla p$ and $\vec{v} \cdot \nabla \rho$, which appear in the resulting expressions, are disregarded (since they are products of perturbation quantities) the simplified equations are

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + \vec{V} \cdot \nabla p + \vec{v} \cdot \nabla P \quad \text{and} \quad \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \vec{v} \cdot \nabla \rho_s \ . \tag{17}$$

It is now assumed that the perturbed pressure and density, p and ρ, vary according to the adiabatic relationship given by Lamb (1945):

$$\frac{dp}{d\rho} = c^2 \ , \tag{18}$$

where c is the velocity of sound at height z, as previously defined. Dividing the first of equations (17) by the second, and substituting into the adiabatic relation (18) yields

$$\frac{\frac{dp}{dt}}{\frac{d\rho}{dt}} = \frac{dp}{d\rho} = c^2 = \frac{\frac{\partial p}{\partial t} + \vec{V} \cdot \nabla p + \vec{v} \cdot \nabla P}{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \vec{v} \cdot \nabla \rho_s} . \qquad (19)$$

This rather unwieldy expression can be simplified somewhat by noting that

$$\vec{v} \cdot \nabla P = v_z \frac{\partial P}{\partial z} \quad \text{and} \quad \vec{v} \cdot \nabla \rho_s = v_z \frac{\partial \rho_s}{\partial z} ,$$

since, by assumption, P and ρ_s are functions of z only. Furthermore, defining the operator $\frac{D}{Dt}$ as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V} \cdot \nabla = \frac{\partial}{\partial t} + V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} , \qquad (20)$$

equation (19) becomes

$$c^2 \left(\frac{D\rho}{Dt} + v_z \frac{\partial \rho_s}{\partial z}\right) = \frac{Dp}{Dt} + v_z \frac{\partial P}{\partial z} . \qquad (21)$$

2. Linearized Equations in Standard Notation

It seems advisible at this point to rewrite equations (12), (15) and (21) in terms of the notation which traditionally has been used in the analysis of microbarometric oscillations. Therefore, $\vec{v}$ (given by equation 7) and $\vec{V}$ (given by equation 4) are now redefined as

$$\vec{v} \equiv (v_x, v_y, v_z) \equiv (u, v, w) \quad \text{and} \quad \vec{V} \equiv (V_x, V_y, 0) \equiv (U, V, 0) . \qquad (22)$$

With this change of notation, the operator $\frac{D}{Dt}$, defined by equation (20) becomes

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y} . \qquad (23)$$

The scalar equations of motion for the perturbed state (equations 12) may now be written in the form:

$$\frac{\partial p}{\partial x} = -\rho_s\left(\frac{Du}{Dt} + U' w\right) , \qquad (24)$$

$$\frac{\partial p}{\partial y} = -\rho_s\left(\frac{Dv}{Dt} + V' w\right) , \qquad (25)$$

$$\frac{\partial p}{\partial z} = -\rho_s\left(\frac{Dw}{Dt} + \frac{g}{\rho_s}\rho\right) , \qquad (26)$$

where a prime (') denotes partial differentiation with respect to z.

Numerous investigators have derived similar equations. However, the appearance of wind shear terms, U' and V', in the linearized equations of motion is unique in the analytical treatment of microbarometric oscillations. In earlier studies, the undisturbed wind velocity was assumed to be either zero at all heights (Lamb, 1911) or a constant value at all heights below a surface of discontinuity (Goldie, 1925; Namekawa, 1934; and others). At the interface, the wind velocity was permitted to change discontinuously to a new constant value in the upper stratum. While Martyn's (1950) mathematical analysis of microbarometric oscillations allows a variation of the undisturbed wind speed with height, his original linearized equations of motion do not contain wind shear terms. The reason for this omission remains obscure.

Defining the divergence of the perturbation velocity by

$$\chi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} , \qquad (27)$$

the equation of continuity (equation 15) may be expressed as

$$\frac{D\rho}{Dt} + \rho_s' w = -\rho_s \chi . \qquad (28)$$

Lamb (1911) was the first author to describe the wave motion associated with microbarometric oscillations in terms of the divergence, χ, of the velocity perturbation and most subsequent studies have followed this procedure. While it is possible to formulate the analysis directly in terms of the pressure perturbation p, a more concise exposition results from the use of the divergence, at least in the Eulerian system of notation.

The adiabatic relationship (equation 21) assumes the form

$$c^2\left(\frac{D\rho}{Dt} + \rho_s' w\right) = \frac{Dp}{Dt} - \rho_s g w , \tag{29}$$

where $\frac{\partial P}{\partial z}$ has been replaced by the equivalent term $-\rho_s g$, given by the hydrostatic equation (3). Equations (28) and (29) may be combined to give

$$\frac{Dp}{Dt} = \rho_s c^2\left(\frac{g}{c^2} w - \chi\right) . \tag{30}$$

Equations (24), (25), (26), (28) and (30) form the basic set of relationships to be used in the following analysis of microbarometric oscillations. This set of equations contains, essentially, five unknown quantities: the pressure perturbation p; the density perturbation ρ; and the three components of the velocity perturbation, u,v,w. Since the quantities $U, U', V, V', \rho_s, \rho_s'$ and c may be determined from an upper air sounding, they are considered known values, along with g which is assumed to be a constant.

3. Solutions for u,v,w

Eliminating the pressure perturbation p and the density perturbation ρ from the basic set of equations yields three equations in the unknown components u,v,w of the velocity perturbation:

$$\frac{D^2u}{Dt^2} - V\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) + V\frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) + K\frac{\partial w}{\partial x} + U'\frac{\partial w}{\partial t} = c^2\frac{\partial \chi}{\partial x} , \tag{31}$$

$$U\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) + \frac{D^2v}{Dt^2} - U\frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) + K\frac{\partial w}{\partial y} + V'\frac{\partial w}{\partial t} = c^2\frac{\partial \chi}{\partial y} , \tag{32}$$

$$- g \frac{\partial u}{\partial x} + U' \frac{Du}{Dt} - g \frac{\partial v}{\partial y} + V' \frac{Dv}{Dt} + \frac{D^2 w}{Dt^2} + (U'^2 + V'^2)\, w =$$

$$c^2 \chi' + \frac{\rho_s'}{\rho_s} c^2 \chi + \frac{\partial c^2}{\partial z} \chi \,, \qquad (33)$$

where $K \equiv U U' + V V' + g$, and a prime (') denotes differentiation with respect to z.

Since the coefficients in equations (31) through (33) are functions of z only, the equations may be solved by the method of separation of variables by assuming that

$$u, v, w, \chi \sim e^{i(k_x x + k_y y - \sigma t)} \,, \qquad (34)$$

where k_x and k_y are the horizontal wave numbers $(\frac{2\pi}{\lambda})$ in the x- and y-directions and σ is the angular frequency ($\frac{2\pi}{T}$, where T is the period). The specific dependence of u, v, w and χ on z remains to be established. The operator $\frac{D}{Dt}$, defined by equation (23), when applied to quantities assumed to vary in accordance with equation (34), becomes:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y} = - i\sigma + i\, k_x U + i\, k_y V = - i(\sigma - k_x U - k_y V).$$

Defining $f \equiv \sigma - k_x U - k_y V$ gives

$$\frac{D}{Dt} \begin{bmatrix} u \\ v \\ w \\ \chi \end{bmatrix} = - i\, f \begin{bmatrix} u \\ v \\ w \\ \chi \end{bmatrix}. \qquad (35)$$

Equations (31) through (33) then reduce to the matrix equation:

$$\begin{bmatrix} -f^2 - fk_y V & fk_x V & i(k_x K - \sigma U') \\ fk_y U & -f^2 - fk_x U & i(k_y K - \sigma V') \\ gk_x + fU' & gk_y + fV' & i(-f^2 + U'^2 + V'^2) \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = ic^2 \begin{bmatrix} k_x \chi \\ k_y \chi \\ \chi' - \frac{1}{H} \chi \end{bmatrix} , \qquad (36)$$

since it may be shown (from equation 5) that

$$\frac{\rho_s'}{\rho_s} + \frac{1}{c^2} \frac{\partial c^2}{\partial z} = - \frac{1}{H} \,.$$

Analytic expressions for u, v, w, will be written in terms of the scale height H for convenience. From equation (36), after considerable simplification, the components of the perturbation velocity may be expressed, assuming $f \neq 0$, as:

$$iM\,u = \left[k_x(f^2 - \frac{g}{H}) + (\frac{f}{H} + f')U' + k_y \frac{g}{f}(k_xV' - k_yU')\right]\chi + (k_xg - fU')\,\chi', \quad (37)$$

$$iM\,v = \left[k_y(f^2 - \frac{g}{H}) + (\frac{f}{H} + f')V' + k_x \frac{g}{f}(k_xV' - k_yU')\right]\chi + (k_yg - fV')\,\chi', \quad (38)$$

$$-M\,w = (k^2g - \frac{f^2}{H} - f\,f')\,\chi + f^2\,\chi', \quad (39)$$

where $M = \dfrac{f^4 - k^2g^2}{c^2}$, $k^2 = k_x^2 + k_y^2$, and $f' = \dfrac{\partial f}{\partial z} = -(k_xU' + k_yV')$.

Substitution of these values for u,v,w, into equation (36) reduces the equation to an identity, showing that the above expressions for the components of the perturbation velocity represent a valid solution.

4. Basic Differential Equation in χ

Taking $\frac{\partial}{\partial x}$ on equation (37), $\frac{\partial}{\partial y}$ on equation (38) and adding yields

$$M(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = \left[f^2k^2 - f'^2 - (\frac{gk^2 + f\,f'}{H})\right]\chi + (gk^2 + f\,f')\,\chi'\,. \quad (40)$$

Taking $\frac{\partial}{\partial z}$ on equation (39) gives

$$M(\frac{\partial w}{\partial z}) = -f^2\chi'' - \left[gk^2 - \frac{f^2}{H} + f\,f' + f^2(\frac{-4f^3f'}{Mc^2} + \frac{H'}{H})\right]\chi'$$
$$+ \left[f\,f' + f'^2 + 2f\,\frac{f'}{H} - f^2\,\frac{H'}{H^2} + (\frac{-4f^3f'}{Mc^2} + \frac{H'}{H})(-gk^2 + \frac{f'}{H} + ff')\right]\chi. \quad (41)$$

Adding equations (40) and (41) yields

$$M\,\chi = -f^2\chi'' - f^2(\frac{H' - 1}{H} - \frac{4f^3f'}{Mc^2})\,\chi' - f^2\left[-k^2 + \frac{gk^2}{f^2H} - \frac{f'}{fH} - \frac{f''}{f} + \frac{H'}{H^2}\right.$$
$$\left. + (\frac{H'}{H} - \frac{4f^3f'}{Mc^2})(\frac{gk^2}{f^2} - \frac{1}{H} - \frac{f'}{f})\right]\chi\,. \quad (42)$$

Let the dimensionless quantity $\frac{4f^4}{f^4 - k^2g^2} \equiv r$ (assuming $f^2 \neq \pm kg$), so that

$$\frac{4f^3f'}{Mc^2} = \frac{4f^4}{f^4 - k^2g^2}\frac{f'}{f} = r\frac{f'}{f}. \tag{43}$$

Then, after considerable simplification, equation (42) assumes the form:

$$\chi'' + \left[\frac{H' - 1}{H} - r\frac{f'}{f}\right]\chi' + \left[-k^2 + \frac{f^2}{c^2} + \frac{gk^2}{f^2}\left(\frac{H' + 1}{H} - \frac{1}{\gamma H} - r\frac{f'}{f}\right) - \frac{f''}{f} - \frac{f'}{f}\left(\frac{H' + 1}{H} - \frac{r}{H}\right) + r\left(\frac{f'}{f}\right)^2\right]\chi = 0, \tag{44}$$

provided that $f \neq 0$.

A number of simplified forms of equation (44) have appeared in the literature, although in none of these cases has the complete equation given here been presented. Namekawa (1934), in his treatment of interface waves, made the initial assumption that the undisturbed wind was constant in magnitude and direction at all heights below the interface. This assumption implies that $f' = 0$ in equation (44) and this simplification reduces equation (44) to Namekawa's basic equation. Martyn (1950) considered the more general case of a wind variable with height, but his analytical solution is given only for the condition of zero wind at all heights. Martyn's equation follows directly from equation (44) by letting $f = \sigma$ and $f' = 0$. Perhaps the ultimate simplification of equation (44) is that given by Friedman (1966), who assumed both an isothermal temperature structure ($H' = 0$) and the absence of undisturbed winds ($f = \sigma$ and $f' = 0$).

In each of these approximations to equation (44) the effect of wind shear has been eliminated. While this approach permits a more rigorous mathematical treatment, the atmospheric wind structure so considered is completely unrealistic. On the other hand, the variable and involved coefficients in equation (44) make an exact solution of this equation over deep strata of the atmosphere impossible.

Accurate wave solutions, within the realistic wind structure important for the study of microbarometric oscillations, may be obtained by treating the atmosphere as a series of thin horizontal layers. In each layer appropriate mean values of the temperature and temperature gradient and the wind velocity and wind velocity gradient will be selected and assumed to be constant. It is admitted that this procedure is a convenient fiction to allow the coefficients of χ' and χ in equation (44) to be taken as constants.

Designating the (assumed) constant coefficient of the χ' term by $-2m$ and the (assumed) constant coefficient of the χ term by a^2, equation (44) may be written for any layer in which $f \neq 0$ as

$$\chi'' - 2m\ \chi' + a^2 \chi = 0 \ . \tag{45}$$

In equation (34) the divergence χ was assumed to be proportional to an exponential term. Thus, by defining

$$E \equiv e^{mz}\ e^{i(k_x x + k_y y - \sigma t)}$$

and

$$k_z^2 \equiv m^2 - a^2 \ , \tag{46}$$

three solutions of equation (45) may be written:

$$\chi_i = (A_i\ e^{k_z z} + B_i\ e^{-k_z z})E \qquad \text{if } k_z^2 > 0 \ , \tag{47}$$

$$\chi_o = (A_o + B_o z)E \qquad \text{if } k_z = 0 \ , \tag{48}$$

$$\chi_c = (A_c\ e^{ik_z z} + B_c\ e^{-ik_z z})E \qquad \text{if } k_z^2 < 0 \ , \tag{49}$$

where A and B, with subscripts i, o and c, are arbitrary constants.

The symbol k_z, defined by equation (46), may be identified as the vertical wave number. The solutions represented by equations (47) and

(48) are non-periodic in the vertical direction. Since equation (47) may be expressed in the form

$$\chi_i \sim E\, e^{\pm k_z z} \quad ,$$

this solution represents an interface wave.

On the other hand, equation (49) may be written

$$\chi_c \sim E\, e^{\pm i k_z z} \; ,$$

where the quantity E contains the factor e^{mz}, so that this solution represents a damped (if $m < 0$) or exponentially growing (if $m > 0$) cellular type of internal wave. The intermediate case, where $k_z = 0$ (equation 48) occurs at levels in the atmosphere where the wave motion changes from an interface to a cellular mode of propagation.

Since the arbitrary constants A and B in equations (47) through (49) depend on the character of the solutions as well as upon appropriate boundary conditions, these constants were designated with subscripts i (for interface waves), o (for the transitional phase where $k_z = 0$) and c (for cellular waves).

Thus, the type of wave motion resulting from equation (44) is determined by the real or imaginary nature of the vertical wave number k_z. Since k_z may be specifically evaluated in terms of measurable parameters, a criterion is established which permits determination of the type of wave motion possible in any atmospheric layer in which $f \neq 0$.

5. <u>Boundary Conditions and Layer Connection Formulas</u>

There are two fundamental continuity requirements in any region of the atmosphere containing wave motion. These requirements are that the vertical component, w, of the velocity perturbation and $\frac{Dp}{Dt}$ must be continuous at all heights. From equation (30), the continuity of w and $\frac{Dp}{Dt}$ assures the continuity of the divergence χ at all heights. Since w and χ are continuous functions, they form convenient variables for the study of microbarometric oscillations. The continuity requirements may be expressed concisely as

$$w_n(z_{n-1}) = w_{n-1}(z_{n-1}) \quad \text{and} \quad \chi_n(z_{n-1}) = \chi_{n-1}(z_{n-1}) \ , \qquad (50)$$

where the subscript n denotes the n^{th} layer, with upper boundary z_n and lower boundary z_{n-1}.

Combining each of the χ solutions (equations 47 through 49) with equation (39) for w yields three forms of the w equation:

$$w = \left[\epsilon A_i \ e^{k_z z} + \delta B_i \ e^{-k_z z}\right] E \qquad \text{if } k_z^2 > 0 \ , \qquad (51)$$

$$w = h\left[A_o + (z - \frac{f^2}{hM}) \ B_o\right] E \qquad \text{if } k_z = 0 \ , \qquad (52)$$

$$w = \left[\epsilon A_c \ e^{ik_z z} + \delta B_c \ e^{-ik_z z}\right] E \qquad \text{if } k_z^2 < 0 \ , \qquad (53)$$

$$\text{where } Mh \equiv \frac{f^2}{H} + f\ f' - (k^2 g + f^2 m), \quad M\epsilon \equiv Mh - f^2 k_z \quad \text{and} \quad M\delta \equiv Mh + f^2 k_z . \qquad (54)$$

Following the general procedure given by Friedman (1966), the arbitrary constants A and B may be eliminated by setting $z = z_{n-1}$ and $z = z_n$ in each of the pairs of equations, (47) and (51), (48) and (52), (49) and (53). The resulting matrix equation:

$$\begin{bmatrix} w_n(z_n) \\ \chi_n(z_n) \end{bmatrix} = \begin{bmatrix} (a_n)_{11} & (a_n)_{12} \\ (a_n)_{21} & (a_n)_{22} \end{bmatrix} \begin{bmatrix} w_n(z_{n-1}) \\ \chi_n(z_{n-1}) \end{bmatrix} \qquad (55)$$

provides the values of w and χ at the upper boundary of any layer n in which $f \neq 0$ in terms of the corresponding values at the lower boundary. The components a_n of the matrix in equation (55) may be expressed:

	$k_z^2 > 0$	$k_z = 0$	$k_z^2 < 0$
$(a_n)_{11}$	$e^{m\theta}(\cosh k_z\theta - b \sinh k_z\theta)$	$e^{m\theta}(1-s\theta)$	$e^{m\theta}(\cos k_z\theta - b \sin k_z\theta)$
$(a_n)_{12}$	$e^{m\theta}(bh - \frac{h}{b})\sinh k_z\theta$	$e^{m\theta}(sh\theta)$	$e^{m\theta}(bh + \frac{h}{b})\sin k_z\theta$
$(a_n)_{21}$	$e^{m\theta}(-\frac{b}{h})\sinh k_z\theta$	$e^{m\theta}(-\frac{s}{h}\theta)$	$e^{m\theta}(-\frac{b}{h})\sin k_z\theta$
$(a_n)_{22}$	$e^{m\theta}(\cosh k_z\theta + b \sinh k_z\theta)$	$e^{m\theta}(1+s\theta)$	$e^{m\theta}(\cos k_z\theta + b \sin k_z\theta)$

(56)

All quantities are appropriate mean values in the $n^{\underline{th}}$ layer, in which $f \neq 0$. The layer width $\theta \equiv z_n - z_{n-1}$, $s \equiv Mh/f^2$ and $b \equiv s/k_z$. In each case, as $\theta \rightarrow 0$, the matrix a_n reduces to the identity matrix. Where k_z is changing rapidly, layers must be selected with very small θ.

Substituting the continuity conditions (50) into matrix equation (55) gives

$$\begin{bmatrix} w_{n+1}(z_n) \\ \chi_{n+1}(z_n) \end{bmatrix} = \begin{bmatrix} (a_n)_{11} & (a_n)_{12} \\ (a_n)_{21} & (a_n)_{22} \end{bmatrix} \begin{bmatrix} w_n(z_{n-1}) \\ \chi_n(z_{n-1}) \end{bmatrix}$$

and subsequently,

$$\begin{bmatrix} w_n(z_{n-1}) \\ \chi_n(z_{n-1}) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} w_1(z_o) \\ \chi_1(z_o) \end{bmatrix} , \qquad (57)$$

where A is the matrix resulting from the product of (n-1) layer matrices. Equation (57) provides the values of w and χ at the lower boundary of the $n^{\underline{th}}$ layer (or at the upper boundary of the (n-1) layer).

The basic boundary condition is that w be zero at the ground, so that $w_1(z_o) = 0$. From equation (57),

$$w_n(z_{n-1}) = A_{12}\chi_1(z_o) \quad \text{and} \quad \chi_n(z_{n-1}) = A_{22}\chi_1(z_o) \; . \qquad (58)$$

Since A_{12} and A_{22} may be evaluated from an atmospheric sounding and $\chi_1(z_o)$ may be determined from measurements of the perturbed pressure at the ground (this will be shown in the next section), w and χ may be calculated at any height z.

An upper boundary condition is not specifically required but it may be assumed that the region of the atmosphere of interest extends from the ground to the lowest layer in which the perturbation pressure p vanishes. It will now be shown that $p \rightarrow 0$ in any layer in which $(gw/c^2) \rightarrow \chi$, provided $f \neq 0$ in that layer.

6. The Pressure Perturbation p

The magnitude of the pressure perturbation at the ground is the primary physical measurement obtained during the study of microbarometric oscillations. This value of p is related to the functions w and χ through equation (30):

$$\frac{Dp}{Dt} = \rho_s c^2 \left(\frac{g}{c^2} w - \chi\right) .$$

Expanding $\frac{Dp}{Dt}$ and recalling that $c^2 = \frac{\gamma P}{\rho_s}$ gives

$$\frac{\partial p}{\partial t} + U \frac{\partial p}{\partial x} + V \frac{\partial p}{\partial y} = \gamma P \left(\frac{g}{c^2} w - \chi\right) . \tag{59}$$

Since p is observed to vary sinusoidally, it may be assumed that

$$p \sim e^{i(k_x x + k_y y - \sigma t)} . \tag{60}$$

Performing the indicated operations in equation (59), using the definition of f given by equation (35) and assuming that $f \neq 0$ gives

$$p = \frac{i \gamma P}{f} \left(\frac{g}{c^2} w - \chi\right) . \tag{61}$$

Equation (61) may be derived rigorously from the basic equations without the assumption that p varies according to equation (60). Such a derivation, however, is quite lengthy and the final result is simply equation (61).

At the ground ($z = z_o$) the vertical component, w, of the perturbation velocity vanishes, so that equation (61) at the ground becomes

$$p_1(z_o) = -i\gamma \frac{P_1(z_o)}{f_1(z_o)} \chi_1(z_o) \ . \tag{62}$$

Combining equation (62) with equations (58) shows that

$$w_n(z_{n-1}) = A_{12} \frac{if_1(z_o)}{\gamma P_1(z_o)} p_1(z_o) \tag{63}$$

and

$$\chi_n(z_{n-1}) = A_{22} \frac{if_1(z_o)}{\gamma P_1(z_o)} p_1(z_o) \ . \tag{64}$$

The perturbed pressure in any layer n, in which $f \neq 0$, can now be expressed from equations (61), (63) and (64) as

$$p_n(z_{n-1}) = -\frac{f_1(z_o)}{f_n(z_{n-1})} \frac{P_n(z_{n-1})}{P_1(z_o)} \left[\frac{g}{c_n^2(z_{n-1})} A_{12} - A_{22} \right] p_1(z_o) \ . \tag{65}$$

7. <u>The Vertical Wave Number k_z</u>

In order to evaluate the components (a_n) of matrix (55) it is necessary to know the value of k_z in every layer. From the coefficients of equation (44) and the assumptions immediately preceding equation (45),

$$m \equiv -\tfrac{1}{2}\left(\frac{H' - 1}{H} - r\,\frac{f'}{f}\right) \tag{66}$$

$$a^2 \equiv -k^2 + \frac{f^2}{c^2} + \frac{gk^2}{f^2}\left(\frac{H'+1}{H} - \frac{1}{H} - r\,\frac{f'}{f}\right) - \frac{f'}{f}\left(\frac{H'+1}{H} - \frac{r}{H}\right) - \frac{f''}{f} + r\left(\frac{f'}{f}\right)^2 \ . \tag{67}$$

Using the definition of H (equation 6) and the hydrostatic equation (3), it follows that

$$\frac{gk^2}{f^2}\left(\frac{H' + 1}{H} - \frac{1}{\gamma H}\right) = \frac{k^2}{f^2}\left[-g\left(\frac{\rho_s'}{\rho_s} + \frac{g}{c^2}\right)\right] = \frac{k^2N^2}{f^2} ,$$

in terms of the Väisälä-Brunt frequency, N, defined by equation (1). Equation (67) then becomes

$$a^2 \equiv k^2\left(\frac{N^2}{f^2} - 1\right) + \frac{f^2}{c^2} - \frac{f''}{f} - \frac{f'}{f}\left[\frac{H' + 1}{H} + r\left(\frac{gk^2}{f^2} - \frac{1}{H} - \frac{f'}{f}\right)\right] . \qquad (68)$$

Defining k_z^2 by equation (46), and using equations (66) and (68), yields

$$k_z^2 \equiv \left(\frac{H'-1}{2H} - \frac{rf'}{2f}\right)^2 + k^2\left(1 - \frac{N^2}{f^2}\right) - \frac{f^2}{c^2} + \frac{f''}{f} + \frac{f'}{f}\left[\frac{H'+1}{H} + r\left(\frac{gk^2}{f^2} - \frac{1}{H} - \frac{f'}{f}\right)\right] . \quad (69)$$

After considerable simplification and elimination of the scale height H, equation (69) becomes

$$k_z^2 \equiv k^2\left(1 - \frac{N^2}{f^2}\right) - \frac{f^2}{c^2} + \left(\frac{\rho_s'}{2\rho_s} + \frac{\gamma g}{c^2}\right)^2 + \frac{f'}{f}\left(\frac{f''}{f'} - \frac{\rho_s'}{\rho_s}\right) + \frac{rf'}{f}\left[\frac{\rho_s'}{2\rho_s} + \frac{gk^2}{f^2} - \left(1 - \tfrac{1}{4}r\right)\frac{f'}{f}\right] . \qquad (70)$$

Although equation (70) remains formidable, it provides the complete definition of the vertical wave number k_z in any layer in which $f \neq 0$. No simplifications based on empirical considerations or particular orientations of the coordinate system have, as yet, been introduced. A simplified form of equation (70) will now be derived.

The observed properties of microbarometric oscillations show that the periods and wavelengths are restricted to fairly narrow ranges (see Table I, page 19). For periods between 5 and 10 minutes, $\sigma\,(= \frac{2\pi}{T})$ lies in the interval

$$10^{-2} < \sigma < 2 \times 10^{-2}\ \text{sec}^{-1}.$$

Wavelengths between 0.5 and 20 kilometers correspond to a range of wave numbers k $(= \frac{2\pi}{\lambda})$, given by

$$3 \times 10^{-6} < k < 1.3 \times 10^{-4} \text{ cm}^{-1} .$$

To determine the order of magnitude of f, consider only two dimensions (x,z) and assume that $f \equiv \sigma - k\bar{u}$, where $\bar{u}$ is the component of the undisturbed wind in the direction of wave propagation. Then typical values of $k\bar{u}$, for the wave number range given above and for wind speeds between 0.5 and 20 m/sec, lie in the interval

$$1.5 \times 10^{-4} < k\bar{u} < 2.5 \times 10^{-1} \text{ sec}^{-1} .$$

Since $\bar{u}$ may be positive or negative, depending on whether the wave motion is propagating with or against the wind, each value of $k\bar{u}$ gives two values of f. Combining values of $k\bar{u}$ with wave frequencies (σ) corresponding to periods between 5 and 10 minutes shows that

$$-0.25 < f < 0.25 \text{ sec}^{-1} , \qquad (71)$$

so that $\frac{kg}{f^2} > 0.05$.

The dimensionless quantity r, defined by equation (43), may be written as

$$r = \frac{4}{1 - (\frac{kg}{f^2})^2} . \qquad (72)$$

As $\frac{kg}{f^2} \rightarrow 1$, $r \rightarrow \infty$ (and $M \rightarrow 0$). But $\frac{kg}{f^2} \rightarrow 1$ implies that

$$\bar{u} \approx \bar{c} \pm \sqrt{\frac{g}{k}} ,$$

where $\bar{c}$ is the wave speed with respect to the ground. For wavelengths between 0.5 and 20 km

$$30 < \sqrt{\frac{g}{k}} < 100 \text{ m/sec}.$$

Since $\bar{c} \approx 10$ m/sec, r becomes large only at positive wind speeds of the order of 40 to 100 m/sec (unlikely in the lower troposphere) or negative wind speeds of the order of -90 to -20 m/sec. A negative wind implies wave motion propagating against the wind, so that this last range of values has little relevance to the study of microbarometric oscillations. It appears reasonable to assume that $f^2 \neq kg$, $r \neq \infty$ and $M \neq 0$ for all subsequent analyses.

Wave disturbances associated with microbarometric oscillations usually have phase velocities very similar to normal tropospheric wind velocities. Under these conditions, $\bar{u} \approx \bar{c}$, $f \approx 0$ (the case where $f = 0$ will be discussed in the next section), so that $r \approx 0$. For example, if

$$\sigma = 1.4 \times 10^{-2}\ \text{sec}^{-1}\ (\sim T = 7.5\ \text{min})$$

$$k = 1.3 \times 10^{-5}\ \text{cm}^{-1}\ \ (\sim \lambda = 5\ \text{km}) \qquad (73)$$

and $\bar{u} = 5$ m/sec,

simple calculations show that

$$\bar{c} \approx 11\ \text{m/sec},\ f = 0.77 \times 10^{-2}\ \text{sec}^{-1},\ \frac{kg}{f^2} \approx 210 \text{ and } r \approx -9 \times 10^{-5}.$$

In general, if $\frac{kg}{f^2} > 20$, $r < (-0.01)$ and the larger $\frac{kg}{f^2}$ becomes (as $\bar{u} \to \bar{c}$), the smaller the value of r. Due to the possible wide variance in the value of r, it should be explicitly calculated whenever the wind speed is significantly different from the wave speed.

To determine if r can be disregarded, it is necessary to compare the order of magnitude of r with the probable magnitude of the other terms in the basic differential equation (44). Using the numerical values given in (73), along with typical values of the temperature and temperature gradient and making the plausible assumption that $\bar{u}' < 10^{-2}$ sec^{-1}, shows that

$$\frac{H' + 1}{H} \approx 10^{-6}\ \text{cm}^{-1},\ \frac{rf'}{f} \approx 10^{-9}\text{cm}^{-1},\ \frac{1}{\gamma H} \approx 10^{-6}\text{cm}^{-1},\ \frac{r}{H} \approx 10^{-10}\ \text{cm}^{-1},$$

$$k^2 \approx 10^{-10}\ \mathrm{cm}^{-2},\ \frac{f^2}{c^2} \approx 10^{-13}\mathrm{cm}^{-2} \text{ and } r(\frac{f'}{f})^2 \approx 10^{-14}\mathrm{cm}^{-2}.$$

Therefore, the following simplifications appear to be justified:

χ' – term: $\quad \frac{H'-1}{H} - r\frac{f'}{f} \approx \frac{H'-1}{H}$;

χ – term:

$$\frac{gk^2}{f^2}(\frac{H'+1}{H} - \frac{1}{\gamma H} - r\frac{f'}{f}) \approx \frac{gk^2}{f^2}(\frac{H'+1}{H} - \frac{1}{\gamma H}) = \frac{k^2 N^2}{f^2};$$

$$-\frac{f'}{f}(\frac{H'+1}{H} - \frac{r}{H}) \approx -\frac{f'}{f}(\frac{H'+1}{H}) = \frac{f'}{f}\frac{\rho_s'}{\rho_s};$$

and disregard $\frac{f^2}{c^2}$ and $r(\frac{f'}{f})^2$ with respect to k^2.

Equation (44) then reduces to

$$\chi'' + (\frac{H'-1}{H})\chi' + \left[k^2(\frac{N^2}{f^2} - 1) - \frac{f'}{f}(\frac{f''}{f'} - \frac{\rho_s'}{\rho_s})\right]\chi = 0. \qquad (74)$$

For this equation, the simplified vertical wave number, $(k_z)_o$, valid when $r \approx 0$, becomes

$$(k_z^2)_o = k^2(1 - \frac{N^2}{f^2}) + (\frac{\rho_s'}{2\rho_s} + \frac{\gamma g}{c^2})^2 + \frac{f'}{f}(\frac{f''}{f'} - \frac{\rho_s'}{\rho_s}). \qquad (75)$$

If calculations indicate that r cannot be disregarded, the complete vertical wave number, given by equation (70), may be written as

$$k_z^2 = (k_z^2)_o - \frac{f^2}{c^2} + r\frac{f'}{f}\left[\frac{\rho_s'}{2\rho_s} + \frac{gk^2}{f^2} - (1 - \tfrac{1}{4}r)\frac{f'}{f}\right]. \qquad (76)$$

The quantity k_z remains finite except as $f^2 \to kg$, a case excluded by previous assumption, and as $f \to 0$. The conditions arising as $f \to 0$ will now be discussed.

8. <u>The Case of $f = 0$</u>

All equations and relationships presented thus far were derived under the supposition that $f \neq 0$. In the event that f is determined to vanish in the region of the atmosphere of interest, it is necessary to examine the conditions which result at this level (or levels). It is very important to allow $f \rightarrow 0$ at the proper point in the analytical treatment for otherwise, vital terms may be lost. Observe that $f = 0$ does not imply that $f' = 0$.

As $f \rightarrow 0$, equation (42) becomes

$$M\chi = \left(-\frac{gk^2}{H} - gk^2\frac{H'}{H}\right)\chi \quad \text{and } M \rightarrow -\frac{k^2g^2}{c^2}$$

This relation may be expressed

$$k^2\left[g\left(\frac{H'+1}{H} - \frac{g}{c^2}\right)\right]\chi = 0 \quad \text{or} \quad k^2N^2\chi = 0\ , \tag{77}$$

using the definitions of N^2 (equation 1) and H (equation 6) in conjunction with the hydrostatic equation (3). If a wave motion is present in the layer where $f = 0$, then $k^2 \neq 0$. Thus, equation (77) shows that either the temperature gradient is adiabatic ($N^2 = 0$) at the level where $f = 0$ or the divergence (χ) of the velocity perturbation vanishes at this level. A similar result was obtained from an entirely different approach by Sekera (1948).

As $f \rightarrow 0$, equation (35) shows that

$$\frac{Du}{Dt} = \frac{Dv}{Dt} = \frac{Dw}{Dt} = \frac{D\chi}{Dt} = 0.$$

Equations (24) and (25) become

$$\frac{\partial p}{\partial x} = -\rho_s U'w \quad \text{and} \quad \frac{\partial p}{\partial y} = -\rho_s V'w\ ,$$

and assuming that p varies exponentially, as indicated by equation (60), it follows that

$$p = i\rho_s w\frac{U'}{k_x} = i\rho_s w\frac{V'}{k_y}\ .$$

The last result implies that

$$(k_x V' - k_y U')w = 0 , \tag{78}$$

indicating that at the f = 0 level either $k_x V' = k_y U'$ or $w = 0$. If the former condition holds, it follows that

$$f' = -(k_x U' + k_y V') = -\frac{U'}{k_x} k^2 = -\frac{V'}{k_y} k^2 \text{ so that } -\frac{f'}{k^2} = \frac{U'}{k_x} = \frac{V'}{k_y} ,$$

while if $w = 0$, the last equation on page 60 shows that $p = 0$.

Consider the case of an adiabatic temperature gradient at the $f = 0$ level, so that $N^2 = 0$, but assuming $k^2 \neq 0$ and $\chi \neq 0$. Then as $f \rightarrow 0$ equation (39) becomes simply $w = \frac{c^2}{g}\chi$, and since, by assumption, $\chi \neq 0$, it follows that $w \neq 0$, so that $k_x V' = k_y U'$ at the $f = 0$ level. With this result, equations (37) and (38) become, as $f \rightarrow 0$,

$$\frac{u}{k_x} = \frac{v}{k_y} = \frac{ic^2}{gk^2}\left[(\chi' - \frac{1}{H}\chi) - \frac{f'^2}{gk^2}\chi\right] . \tag{79}$$

These values of u, v, w, satisfy the matrix equation (36) under the conditions assumed. Furthermore, as $f \rightarrow 0$, equation (30) becomes $\frac{Dp}{Dt} = 0$ and equation (28) may be written

$$\frac{D\rho}{Dt} = -\rho_{\bar{z}}' \frac{c^2}{g}\chi - \rho_{\bar{z}}\chi = -\rho_{\bar{z}}\frac{c^2}{g}\chi\left(\frac{\rho_{\bar{z}}'}{\rho_{\bar{z}}} + \frac{g}{c^2}\right) = \rho_{\bar{z}}\frac{c^2}{g^2}\chi N^2 = 0.$$

From equations (24) and (25), and with the condition $k_x V' = k_y U'$, it follows that

$$p = -i\frac{\rho_{\bar{z}} f'}{k^2}\frac{c^2}{g}\chi = -i\frac{\rho_{\bar{z}} f'}{k^2} w \quad \text{and} \quad \frac{\partial p}{\partial t} = \frac{\rho_{\bar{z}} f' \sigma}{k^2} w . \tag{80}$$

Equation (26), as $f \rightarrow 0$, shows that $\frac{\partial p}{\partial z} = -g\rho$. Taking $\frac{\partial}{\partial z}$ on the value of p given in equation (80) and equating the result to $-g\rho$ reveals that

$$\rho = -\frac{p}{g\chi}(\chi' - \frac{1}{H}\chi), \text{ or } \frac{p'\chi}{p} = \chi' - \frac{1}{H}\chi.$$

Thus,

$$\frac{u}{k_x} = \frac{v}{k_y} = \frac{iw}{k^2}(\frac{p'}{p} - \frac{f'^2}{gk^2}) .$$

In summary, if $k^2 \neq 0$, $\chi \neq 0$ but $N^2 = 0$, as $f \to 0$,

$$\frac{k_x}{k_y} = \frac{u}{v} = \frac{U'}{V'}, \; w = \frac{c^2}{g}\chi, \; p = -i\frac{\rho f'}{k^2} w \text{ and } \frac{Dp}{Dt} = \frac{D\rho}{Dt} = 0. \tag{81}$$

The second case given by equation (77) is that $\chi = 0$, and assuming $k^2 \neq 0$ and $N^2 \neq 0$, as $f \to 0$, equations (37) through (39) become

$$\frac{u}{k_x} = \frac{v}{k_y} = \frac{ic^2}{gk^2}\chi' \text{ and } w = 0.$$

These values of u, v, w, satisfy matrix equation (36), assuming that $f = 0$ and $\chi = 0$. Equations (28) and (30) show that $\frac{Dp}{Dt} = \frac{D\rho}{Dt} = 0$, and from equations (24) through (26) it follows that $p = \rho = 0$. Thus, if $k^2 \neq 0$ and $N^2 \neq 0$,

$$\frac{k_x}{k_y} = \frac{u}{v}, \; w = 0, \; p = 0, \; \chi = 0 \text{ and } \frac{Dp}{Dt} = \frac{D\rho}{Dt} = 0. \tag{82}$$

Consider the general expression for the pressure perturbation in any layer in which $f \neq 0$ (equation 61). In the first case ($N^2 = 0$), since $w \to \frac{c^2}{g}$ as $f \to 0$, both the numerator and denominator of equation (61) go to zero as $f \to 0$, permitting a finite value of p (given by equation 81). In the second case ($\chi = 0$), as $f \to 0$, both w and χ become zero, again showing that p assumes a definite value (which is zero in this case).

As $f \to 0$, $r \to 0$, and the vertical wave number k_z becomes the limit of equation (75):

$$(k_z^2)_o = \lim_{f \to 0} \left[k^2(1 - \frac{N^2}{f^2}) + (\frac{\rho'}{2\rho_s} + \frac{\gamma g}{c^2})^2 + \frac{f'}{f}(\frac{f''}{f'} - \frac{\rho'}{\rho_s}) \right].$$

In the interests of further simplifying an extremely complex problem, two assumptions will now be made. The validity of the final conclusions is limited to the special situation in which the following assumptions hold:

(a) It is assumed that as $f \to 0$, the term $\left(\frac{\rho_s'}{2\rho_s} + \frac{\gamma g}{c^2}\right)^2$ is negligible with respect to the term $k^2\left(1 - \frac{N^2}{f^2}\right)$.

(b) Furthermore, as $f \to 0$, the term $\frac{f'}{f}\left(\frac{f''}{f'} - \frac{\rho_s'}{\rho_s}\right)$ is assumed to be negligible with respect to $k^2\left(1 - \frac{N^2}{f^2}\right)$.

Assumption (a) is probably true in most cases. In assumption (b), the presence of f in the denominator of the term $\frac{f'}{f}\left(\frac{f''}{f'} - \frac{\rho_s'}{\rho_s}\right)$ prohibits any easy assurance for the validity of neglecting this term as $f \to 0$. How generally the conclusions of this study continue to apply, at least qualitatively, has thus not been here investigated. Interesting topics for further study would be to establish under what conditions neglect of this term is realistic, and also, if not negligible, how its presence affects the final conclusion.

Subsequent discussion will be based on the equation

$$(k_z^2)_o = k^2 \lim_{f \to 0} \left(1 - \frac{N^2}{f^2}\right) . \tag{83}$$

The following values of $(k_z^2)_o$ are possible, depending on the value of N^2 and on the way in which $f \to 0$:

$$\text{If } N^2 > 0,\ (k_z^2)_o \to -\infty \text{ while if } N^2 < 0,\ (k_z^2)_o \to +\infty. \tag{84}$$

$$\text{If } N^2 = 0 \text{ and if } \lim_{f \to 0} \frac{N^2}{f^2} \begin{cases} >1, & \text{then } (k_z^2)_o < 0, \\ \to 1, & \text{then } (k_z)_o = 0, \\ <1, & \text{then } (k_z^2)_o > 0. \end{cases} \tag{85}$$

It is necessary to join the layer in which the wind speed equals the wave speed ($f = 0$) with those layers above and below in which $f \neq 0$. Each of the matrix components a_n in equation (56) becomes infinite as $f \to 0$, for at this level, $s \to \infty$ and $b \to \infty$. In order for these components to remain finite, the following conditions are necessary:

$$\text{If } (k_z^2)_o > 0, \ \sinh k_z\theta \to 0 \text{ as } f \to 0, \ \Rightarrow k_z\theta \to 0, \tag{86}$$

$$(k_z)_o = 0, \quad \theta \to 0 \text{ as } f \to 0,$$

$$(k_z^2)_o < 0, \ \sin k_z\theta \to 0 \text{ as } f \to 0, \ \Rightarrow k_z\theta \to 0, \pi, 2\pi, \ldots \tag{87}$$

Each of these conditions may be satisfied if $\theta \to 0$ as $f \to 0$, but this implies an infinitely slow approach to the level where $f = 0$ ($\theta \to 0$ suggests that the number of layers $\to \infty$). On the other hand, if $k_z \to 0$ but remains real-valued, equation (86) indicates that a connection can be made at the level where $f = 0$. Equation (87) shows that if $(k_z)_o \to 0, \frac{\pi}{\theta}, \frac{2\pi}{\theta}, \ldots$, as $f \to 0$, a connection can be made to the level where $f = 0$, provided that the wave motion is of a cellular type.

Combining conditions (84) with conditions (86) and (87) indicates that in regions where $N^2 \neq 0$ at the level where $f \to 0$, $(k_z^2)_o \to \pm\infty$ and the requirement that $k_z \to 0$ cannot be satisfied. Thus, where $N^2 \neq 0$, the $f = 0$ layer functions as a boundary, across which wave solutions and vertical perturbation motions are not transmitted. This is in complete agreement with equation (82) which shows, if $N^2 \neq 0$, $\chi = w = p = 0$ at the level where $f = 0$. Higher modes of cellular wave solutions (equivalent to $(k_z)_o = \frac{\pi}{\theta}, \frac{2\pi}{\theta}, \ldots$) may be transmitted across the $f = 0$ boundary, but conditions (82) must still apply at this level.

On the other hand, conditions (85) show that if $N^2 = 0$, both real and imaginary values of $(k_z)_o$ may approach zero and hence χ and w may be transmitted across the $f = 0$ level. In this case, definite values of χ, w and p may be expected at the height where $f \to 0$, and equation (81) verifies this contention.

9. Interpretation of Results

Horizontal wave motion may exist in any suitably perturbed atmospheric layer. If the wave velocity is different from the undisturbed wind velocity in this layer and in adjoining layers, multilayered regions may be connected and the effects of the disturbance transmitted vertically until the layer (or layers) in which $f = 0$ is encountered. If the special assumptions made on page 63 are valid and if $N^2 \neq 0$ at this height, the $f = 0$ layer forms a boundary which prevents the vertical transfer of wave motion effects. Disturbances which occur above the lowest $f = 0$ level (assuming $N^2 \neq 0$) cannot be transmitted to the ground and so are irrelevant to the study of microbarometric oscillations recorded at the earth's surface.

However, if $N^2 = 0$ at the $f = 0$ level, the effects caused by wave motions occurring in regions above this layer may be connected to the $f = 0$ layer and then transmitted to the ground. The condition $N^2 = 0$ effectively opens a "window" in the $f = 0$ layer. Under these conditions, equations (79) through (81) show that

$$\frac{u}{k_x} = \frac{v}{k_y} = \frac{iw}{k^2}\left[\left(\frac{\chi'}{\chi} - \frac{1}{H}\right) - \frac{f'^2}{gk^2}\right], \quad w = \frac{c^2}{g}\chi \quad \text{and } p = -i\,\frac{\rho f'}{k^2}\,w\ ,$$

at the level where $f = 0$. The pressure perturbation at the $f = 0$ level is seen to be dependent on wind shear, for if $f' = 0$, $p = 0$ and furthermore, $\rho = 0$ and $p' = 0$. The velocity perturbation, however, still has a finite value, determined by the divergence and its dependence on z, so that perturbation wave motion is still present in the $f = 0$ layer. As $f \to 0$

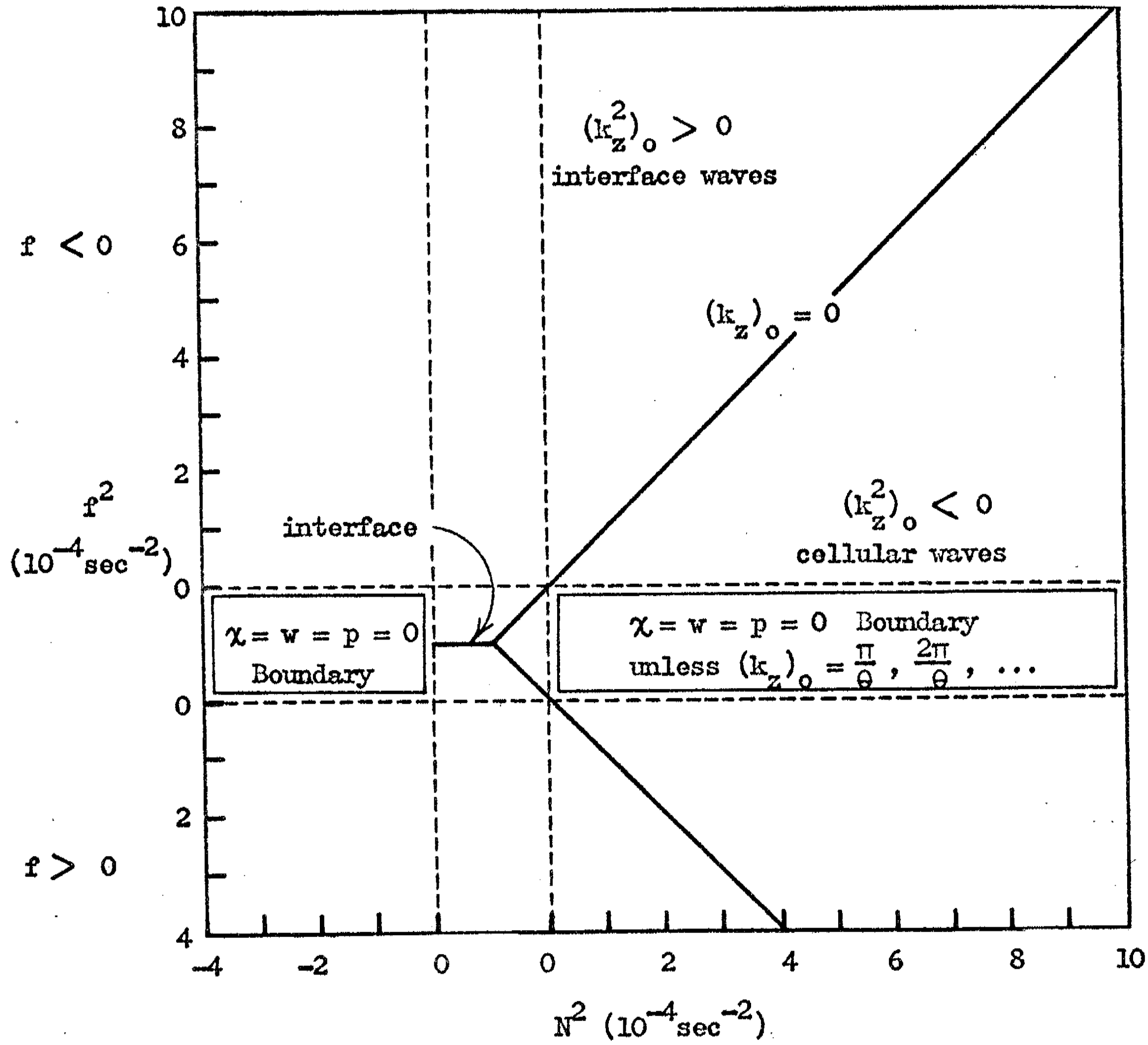

Figure 4. Type of wave motion possible as a function of f^2 and N^2 based on $(k_z^2)_o = k^2(1 - N^2/f^2)$. The $f = 0$ layer and $N^2 = 0$ region have been expanded to show the hypothetical interface and boundary conditions.

with $N^2 = 0$, Figure 4 shows the high probability that $(k_z^2)_o > 0$.

Thus, all the necessary conditions are present for the formation of an interface wave, if the interface is identified as the layer in which $f = 0$ and $N^2 = 0$, rather than being restricted to a physically observable discontinuity in temperature, temperature gradient or wind velocity. The interface wave apparently is caused by a disturbance above or below the interface and modified by the wind shear across the $f = 0$ level. A numerical example of an interface wave case is given in Appendix III.

Returning to the situation of $N^2 \neq 0$ at the level where $f = 0$, the bounded region below this level may contain either interface or cellular waves depending upon the temperature structure and the undisturbed wind velocity. However, if $N^2 < 0$ (superadiabatic lapse rates), organized wave motion is highly unlikely, so it appears that under normal circumstances, $N^2 > 0$ in the region below the $f = 0$ level. Furthermore, since microbarometric oscillations are usually recorded at night, when surface wind speeds are low and inversion temperature gradients ($N^2 \gtrsim 4 \times 10^{-4}$ sec^{-2}) prevalent, Figure 4 indicates that the wave disturbance is probably of the cellular type. The cellular wave appears to be traveling in a duct, bounded above at the level where $f = 0$ (if $(k_z)_o \neq \frac{\pi}{\theta}, \frac{2\pi}{\theta}, \ldots$) and below by the earth's surface. At both of these boundaries the vertical component of the velocity perturbation, w, goes to zero (equation 82).

Martyn (1950) did not describe cellular waves physically, except to suggest that they are "rotational in type". Recent studies, reported by Lumley and Panofsky (1964), indicate that:

> It is thus likely that the atmosphere is generally filled with quasi-horizontal eddies of sizes greater than 2 miles, which very probably derive their energy from horizontal wind shear.

Cellular waves may possibly be related to the motion of such eddies through the quiescent atmosphere. The velocity perturbation introduced by the eddy would probably deform the originally horizontal streamlines of the undisturbed airflow into undulating streamlines. The physical appearance of a typical streamline near the lower boundary of the duct (the ground) would probably be essentially sinusoidal.

These observations on the physical nature of the cellular wave must be considered only speculative at the present time. When complete experimental data is available, computer evaluation of the three components of the velocity perturbation may provide a quantitative basis from which a realistic model of the cellular wave mechanism can be derived.

VII. SUMMARY AND CONCLUSIONS

The present study has been directed specifically to the determination of the most probable mechanism responsible for producing sinusoidal perturbation pressures at the earth's surface. An initial review of the characteristics of microbarometric oscillations indicates that the presence of a tropospheric wave motion is a necessary, but not sufficient, condition for the appearance of pressure fluctuations at the ground. The intervening atmospheric layers must permit the effects of this wave motion to be transmitted to the ground.

Two major types of tropospheric wave motion are apparently capable of producing microbarometric oscillations. The earlier investigators felt that traveling waves on an atmospheric surface of discontinuity provided the most reasonable mechanism. Later, Martyn (1950) suggested that cellular waves were the probable cause of the most commonly recorded perturbations of pressure. It is difficult to understand why the proponents of the interface wave mechanism did not consider any other possibilities, and why Martyn did not consider the possibility of waves on a surface of discontinuity. Since both wave mechanisms lead to essentially identical pressure patterns at ground level, it is necessary to examine the vertical structure of the atmosphere, particularly at the level where the wind and wave velocities become equal, in order to determine which type of wave actually produces a given microbarometric oscillation.

The mathematical analysis of wave motion in a temperature- and wind-stratified atmosphere, as presented here, shows that the type of wave motion possible in any layer, in which $f \neq 0$, depends on the real or imaginary nature of the vertical wave number, k_z, in that layer. Multilayered regions, throughout which $f \neq 0$, may be joined by a connection matrix, permitting the calculation of χ and w, and hence p, at all heights in terms of the measured value of the perturbation pressure at the earth's surface.

At the level where the wind and wave velocities become equal ($f = 0$), a special condition is imposed, requiring either $\chi = 0$ or $N^2 = 0$. In this study, two simplifying assumptions are made at this point, as indicated on page 63. Where these assumptions may be valid, if $\chi = 0$, the $f = 0$ layer forms a boundary preventing the vertical transmission of wave effects. If $N^2 = 0$, the effects of wave motion may be transmitted across the $f = 0$ level.

It seems plausible to identify the $f = 0$ layer, with $N^2 = 0$ and k_z real, as the "discontinuity surface" required for the interface type of wave motion. The importance of wind shear at the $f = 0$ level is demonstrated by the fact that p and ρ both vanish if $f' = 0$.

On the other hand, if k_z remains imaginary in the region between the ground and the $f = 0$ layer, this layer serves as an upper boundary for the surface duct containing the fundamental mode of cellular wave motion. If $k_z = \frac{\pi}{\theta}, \frac{2\pi}{\theta}, \ldots$, where θ is the width of the $f = 0$ layer, higher modes of cellular waves may appear (see Johnson's (1929) remarks on page 8).

Microbarometric oscillations may be caused by interface or cellular wave motions, or by a combination of these motions. However, most sinusoidal pressure variations recorded at ground level occur at night, the number of occurrences decreasing sharply with the onset of morning heating (see Figure 2, page 17). Atmospheric conditions at night are characterized by low values of wind speed and low level temperature inversions. Under these conditions, at the ground,

$$4 \times 10^{-4} sec^{-2} < f^2 < 1 \times 10^{-4} sec^{-2}$$

$$\text{and } N^2 \gtrsim 4 \times 10^{-4} sec^{-2}.$$

For these ranges of f^2 and N^2, Figure 4 shows that $(k_z^2)_o < 0$, indicating a cellular type of wave motion.

To the extent that the assumptions made on page 63 are valid, the mathematical analysis presented here, combined with empirical considerations, indicates that the majority of the sinusoidal pressure perturbations recorded by sensitive microbarographs are probably due to the direct influence of cellular waves.

APPENDIX I - THE EQUATION OF MOTION

The hydrodynamic equation of motion is given on page 42 as equation (11):

$$\frac{\partial \vec{v}}{\partial t} + (\vec{V} + \vec{v})\cdot\nabla\ (\vec{V} + \vec{v}) = -\frac{1}{\rho_z+\rho}\nabla(P + p) - (0,0,g)\ .$$

Multiplying by $(\rho_z+\rho)$ and expanding the left-hand side gives

$$(\rho_z+\rho)\left(\frac{\partial \vec{v}}{\partial t} + \vec{V}\cdot\nabla\vec{V} + \vec{V}\cdot\nabla\vec{v} + \vec{v}\cdot\nabla\vec{V} + \vec{v}\cdot\nabla\vec{v}\right)$$

$$= -\nabla P - \nabla p - (0,0,\rho_z g) - (0,0,\rho g)\ .$$

Rearranging, the equation of the total motion becomes

$$\left[\rho_z\vec{V}\cdot\nabla\vec{V} + \nabla P + (0,0,\rho_z g)\right] + \rho_z\left[\frac{\partial \vec{v}}{\partial t} + \vec{V}\cdot\nabla\vec{v} + \vec{v}\cdot\nabla\vec{V} + \underline{\underline{\vec{v}\cdot\nabla\vec{v}}}\right]$$

$$+\rho\left[\underline{\underline{\frac{\partial \vec{v}}{\partial t}}} + \vec{V}\cdot\nabla\vec{V} + \underline{\underline{\vec{V}\cdot\nabla\vec{v}}} + \underline{\underline{\vec{v}\cdot\nabla\vec{V}}} + \underline{\underline{\vec{v}\cdot\nabla\vec{v}}}\right] = -\nabla p - (0,0,\rho g)\ . \qquad \text{(I-1)}$$

Applying the standard methods of perturbation theory to this equation, it is assumed that terms of second-order in the perturbation quantities ($\vec{v}$, p, ρ) may be disregarded. Thus, the following terms (underlined above) may be omitted from equation (I-1):

$$\vec{v}\cdot\nabla\vec{v},\ \rho\frac{\partial \vec{v}}{\partial t},\ \rho\vec{V}\cdot\nabla\vec{v}\ \text{ and }\ \rho\vec{v}\cdot\nabla\vec{V}\ .$$

Observe that equation (9), page 41, must also apply to the undisturbed state of the atmosphere, so that

$$\vec{V}\cdot\nabla\vec{V} = -\frac{1}{\rho_z}\nabla P - (0,0,g)\ . \qquad \text{(I-2)}$$

Thus, the entire first bracket of equation (I-1) is identically zero. Furthermore, P is assumed to be a function of z only so that

$$\nabla P = (0,0,\frac{\partial P}{\partial z}) = (0,0,\ -\rho_z g)\ ,$$

from the hydrostatic equation (equation 3, page 37). Substitution of this value of ∇P into equation (I-2) shows that $\vec{V}\cdot\nabla\vec{V} = (0,0,0)$, so the entire third bracket of equation (I-1) may be disregarded.

Thus, equation (I-1) reduces to

$$\frac{\partial\vec{v}}{\partial t} + \vec{V}\cdot\nabla\vec{v} + \vec{v}\cdot\nabla\vec{V} = -\frac{1}{\rho_s}\nabla p - (0,0,\frac{g}{\rho_s}\rho) \;. \qquad \text{(I-3)}$$

The term $\vec{v}\cdot\nabla\vec{V}$ may be evaluated from the definitions of $\vec{V}$ (equation 4) and $\vec{v}$ (equation 7), with the result that

$$\vec{v}\cdot\nabla\vec{V} = (v_z\frac{\partial V_x}{\partial z} \;,\; v_z\frac{\partial V_y}{\partial z} \;,\; 0) \;.$$

Therefore, the vector equation of the perturbed motion becomes

$$\frac{\partial\vec{v}}{\partial t} + \vec{V}\cdot\nabla\vec{v} = -\frac{1}{\rho_s}\nabla p - (v_z\frac{\partial V_x}{\partial z} \;,\; v_z\frac{\partial V_y}{\partial z} \;,\; \frac{g}{\rho_s}\rho) \;. \qquad \text{(I-4)}$$

The three scalar equations of motion, equivalent to the vector equation (I-4) are:

$$\begin{aligned}
\frac{\partial v_x}{\partial t} + V_x\frac{\partial v_x}{\partial x} + V_y\frac{\partial v_x}{\partial y} &= -\frac{1}{\rho_s}\frac{\partial p}{\partial x} - \frac{\partial V_x}{\partial z}v_z \;, \\
\frac{\partial v_y}{\partial t} + V_x\frac{\partial v_y}{\partial x} + V_y\frac{\partial v_y}{\partial y} &= -\frac{1}{\rho_s}\frac{\partial p}{\partial y} - \frac{\partial V_y}{\partial z}v_z \;, \qquad (12) \\
\frac{\partial v_z}{\partial t} + V_x\frac{\partial v_z}{\partial x} + V_y\frac{\partial v_z}{\partial y} &= -\frac{1}{\rho_s}\frac{\partial p}{\partial z} - \frac{g}{\rho_s}\rho \;.
\end{aligned}$$

APPENDIX II - THE EQUATION OF CONTINUITY

The equation of continuity for the compressible atmosphere is given on page 42 as equation (14):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_z + \rho)(\vec{V} + \vec{v}) = 0 \ .$$

This equation may be expanded to give

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho_z \vec{V} + \nabla \cdot \rho_z \vec{v} + \nabla \cdot \rho \vec{V} + \nabla \cdot \rho \vec{v} = 0 \ .$$

Disregarding the last term (a product of perturbation quantities) and subtracting the equation of continuity for the undisturbed atmosphere ($\nabla \cdot \rho_z \vec{V} = 0$) yields

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho_z \vec{v} + \nabla \cdot \rho \vec{V} = 0 \ .$$

This equation may be expanded, with the result

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \vec{v} \cdot \nabla \rho_z + \rho_z \nabla \cdot \vec{v} + \rho \nabla \cdot \vec{V} = 0 \ .$$

To simplify this expression recall that ρ_z is assumed to be a function of z only, so that $\vec{v} \cdot \nabla \rho_z = v_z \frac{\partial \rho_z}{\partial z}$. Furthermore, in accordance with the definitions of $\vec{V}$ (equation 4) and $\vec{v}$ (equation 7),

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad \text{and} \quad \nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + 0 = 0 \ .$$

Therefore, the equation of continuity for the perturbed state becomes:

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + v_z \frac{\partial \rho_z}{\partial z} + \rho_z \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0 \ . \qquad (15)$$

APPENDIX III - A NUMERICAL EXAMPLE

The analysis of an actual case of microbarometric oscillations will now be presented as an application of some of the methods developed in this study. This analysis indicates that the probable source of the pressure perturbation was an interface wave at the $f = 0$ level, located approximately 1.25 km above the ground. Unfortunately, the example discussed here is the only case available to the writer at this time. However, the general method illustrated is also applicable to microbarometric oscillations resulting from cellular wave disturbances.

On 18 April 1963 Fullerton (1964) recorded sinusoidal pressure oscillations of average period 6.0 minutes ($\sigma = 1.74 \times 10^{-2}$ sec^{-1}) during the interval 0200 to 0700 MST. Differences in arrival time at the three stations of a nonlinear array were determined by phase comparison of the pressure records. Calculations indicated that a disturbance of wavelength 3 km ($k = 2.09$ km^{-1}) was moving toward the north (from 180°) with an apparent speed of 8 m/sec.

Upper air temperatures, densities and wind velocities were provided by a radiosonde sounding taken at 0945 MST at Stallion Site on the White Sands Missile Range. Stallion Site is located only a few kilometers southeast of the pressure station network. Although the sounding was taken some three hours after cessation of the microbarometric activity, it may be assumed that tropospheric conditions, above the surface layer at least, were essentially the same as those existing throughout the night.

The sounding data, given at 500 foot intervals, were plotted to an altitude of 2.3 km above the ground. Figure 5 shows the temperature and f, the frequency in the moving medium as derived from wind speed values,

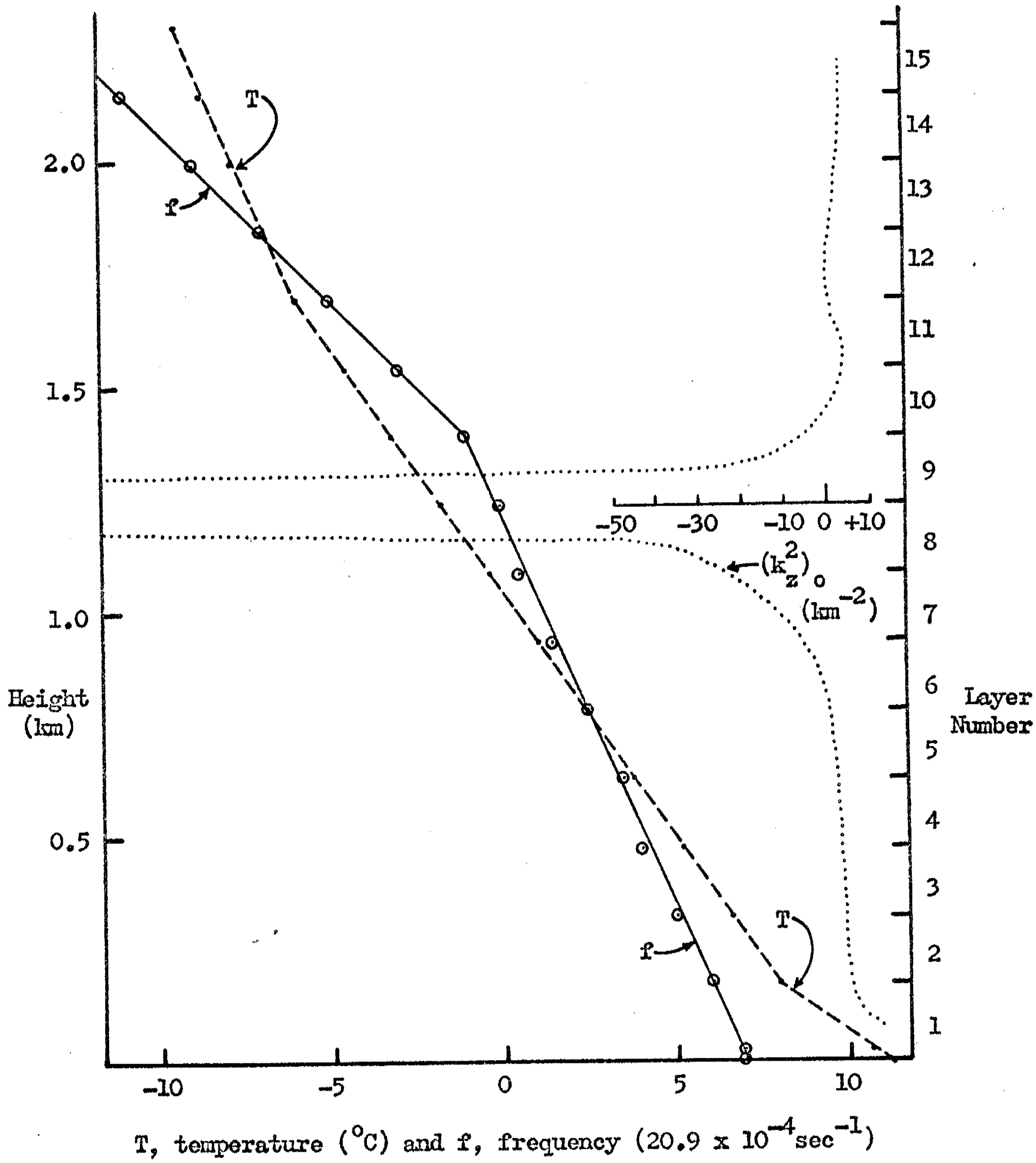

Figure 5. Temperature, frequency in the moving medium and the square of the vertical wave number as functions of height (over 15 layers) for the case of 18 April 1963. The scale inside the right margin is for $(k_z^2)_o$ in km^{-2}. The vertical wave number goes to negative infinity at height 1.25 km (where f = 0). The wave parameters are: $\sigma = 1.74 \times 10^{-2} sec^{-1}$, $k = 2.09\ km^{-1}$, $\bar{c} = 8$ m/sec.

as functions of height. Connecting lines are drawn to give an indication of the temperature and frequency gradients. With the exception of the superadiabatic lapse rate in the surface layer (approximately 170 meters thick), the temperature gradient is reasonably constant, with a slight increase in slope at the 1.7 km level. Wind speeds are somewhat irregular, but a marked increase in the wind speed gradient (decrease in the f gradient) appears near the 1.4 km altitude. The assumption of a sharp discontinuity in the gradient is unrealistic, of course, but it indicates the general level where a fundamental change in wind speed structure occurs.

Density and wind direction data from the sounding were also plotted, although these points are not shown in Figure 5. The density gradient, above the surface layer, was essentially constant. The variation in the wind direction was so small that a constant wind direction of 230° was assumed. There is general agreement between the mean direction of the wind (230°) and the calculated direction of wave propagation (180°). A shift in wind direction of 40° to 60° may well have taken place in the hours between the observed microbarometric activity and the upper air sounding.

The vertical wave number, k_z, was calculated in the surface layer and in 14 additional layers, each approximately 152 meters (500 feet) thick. Since $kg/f^2 > 20$ in each layer, it was possible to use the simplified vertical wave number, $(k_z)_o$, defined by equation (75). Furthermore, calculations show that the term

$$\left(\frac{\rho_z'}{2\rho_z} + \frac{\gamma g}{c^2}\right)^2$$

in equation (75) is negligibly small in each layer and may be disregarded. Table III gives the data necessary for the calculation of $(k_z^2)_o$ and Figure 5 shows the resulting variation in the square of the vertical wave number

TABLE III - Layer Parameters

layer	β	T	N^2	f	f'	$\frac{f'}{f}$	$-\frac{\rho_s'}{\rho_s}$	$k^2(1 - \frac{N^2}{f^2})$	$\frac{f''}{f}$	$-\frac{f'}{f}\frac{\rho_s'}{\rho_s}$	$(k_z^2)_o$
	(^{o}/km)	(^{o}K)	($10^{-4}sec^{-2}$)	($10^{-2}sec^{-1}$)	($10^{-2}sec^{-1}km^{-1}$)	(km^{-1})	(km^{-1})	(km^{-2})	(km^{-2})	(km^{-2})	(km^{-2})
1	18.2	282	-2.91	1.36	-1.21	-0.9	0.055	11.44	0	-0.05	11.39
2	9.2		0.29	1.19		-1.0	0.090	3.52		-0.09	3.43
3				0.98		-1.2		3.08		-0.11	2.97
4				0.79		-1.5		2.33		-0.14	2.19
5				0.61		-2.0		0.97		-0.18	0.79
6		mean		0.44		-2.8		-2.33		-0.25	-2.58
7		274		0.27		-4.5		-13.07		-0.40	-13.47
8				0.08		-15.1	0.090	-195.05	0	-1.36	-196.41
9				-0.10	-1.21	12.1	0.100	-123.20	12.55*	1.21	-109.44
10				-0.42	-2.74	6.5		-2.68	2.99*	0.65	0.96
11	9.2		0.29	-0.84		3.3		2.64	0	0.33	2.97
12	5.8		1.58	-1.25		2.2		-0.04		0.22	0.18
13		mean		-1.67		1.6		1.89		0.16	2.05
14		266		-2.09		1.3		2.82		0.13	2.95
15	5.8		1.58	-2.51	-2.74	1.1	0.100	3.30	0	0.11	3.41

* To compensate for the unrealistic discontinuity in f', f" was calculated over layers 9 and 10 (a thickness of approximately 300 meters) and then divided equally between the two layers.

with height. Detailed calculations of w, χ and p, based on equations (63), (64) and (65), require computer evaluation of the matrix components A_{12} and A_{22}. These calculations were not performed.

The disturbance causing the observed microbarometric activity is clearly not a cellular wave in the surface layers, for $(k_z^2)_o > 0$ in the first 900 meters above the ground. While Table III indicates that $N^2 = 0.29 \times 10^{-4} sec^{-2}$ in layers 8 and 9, an error of less than 1°C. in the temperature sounding, or a slight variation in the temperature occurring between the sounding elevations near the f = 0 level, could easily allow $N^2 \approx 0$ in this region. Therefore, the pressure disturbance has been interpreted as an interface type of wave motion. The interface appears as a relatively shallow layer of high negative $(k_z^2)_o$ values at the height where f = 0, rather than as a physically observable surface of discontinuity. This latter description may also be applied, however, for the f = 0 layer roughly separates regions of different temperature and wind speed structure. Above the interface, $(k_z^2)_o > 0$, preventing the vertical propagation of the wave energy. Thus, the interface wave is essentially concentrated in a rather diffuse transitional zone, within which the wave remains stationary with respect to the moving air.

The superadiabatic temperature gradient in the first layer almost certainly did not exist during the night. If the lapse rate calculated in layers 2 through 11 (9.2 °/km) is assumed to continue to the ground, $(k_z^2)_o > 0$ in the surface layer. If the wind speed had assumed a constant value above 1.4 km, or if the wind shear at this level had been less pronounced, a much thicker stratum of negative $(k_z^2)_o$ values would have resulted. Then the wave energy would not be concentrated at the f = 0 level and the interface would become so diffuse as to vanish. It is probable, under these circumstances, that no microbarometric activity would be observed.

This particular example appears to be an anomaly, for it is the only case of microbarometric oscillations, obtained during a ten month study, in which the disturbance maintained an identifiable waveform over a distance of 10 km.

APPENDIX IV - LIST OF SYMBOLS

Symbol	Explanation	Page	Equation
a^2	(Assumed) constant coefficient of χ term in equation (44)	50 55 56	45 67 68
(a_n)	Components of layer connection matrix (equation 55)	52 53	55 56
$A_{i,o,c}$	Arbitrary constant for interface waves(i), transitional phase(o) or cellular waves(c)	50 52	47-49 51-53
A_{11}	Components of general connection matrix (equation 57)	53	57
b	Symbol for s/k_z	53	—
$B_{i,o,c}$	Arbitrary constant for interface waves(i), transitional phase(o) or cellular waves(c)	50 52	47-49 51-53
c	Velocity of sound (at height z)	37	5
$\bar{c}$	Wave speed with respect to the ground	57	—
c_p	Specific heat at constant pressure of dry air (0.239 cal/gram-deg)	21	—
E	Symbol for $\exp(mz) \exp\left[i(k_x x + k_y y - \sigma t)\right]$	50	—
f	Angular frequency measured in the moving medium	47	35
$\vec{F}_m$	Total frictional force per unit mass	41	8
g	Acceleration due to gravity (10 m/sec^2)	21	—
h	Symbol for $(\frac{f^2}{H} + f f' - k^2 g - f^2 m)/M$	52	54
H	Height of homogeneous atmosphere (scale height)	38	6
k_x	Horizontal wave number in x-direction	47	34
k_y	Horizontal wave number in y-direction	47	34
k	Horizontal wave number $= +\sqrt{k_x^2 + k_y^2}$	48	—
k_z	Vertical wave number	50 56 59	46 70 76
$(k_z)_o$	Simplified vertical wave number	59 63	75 83
K	Symbol for U U' + V V' + g	47	—
m	(Assumed) constant coefficient of $-2\chi'$ term in equation (44)	50 55	45 66
M	Symbol for $(f^4 - k^2 g^2)/c^2$	48	—
n	The n^{th} layer, upper boundary z_n, lower z_{n-1}	52	50

Symbol	Explanation	Page	Equation
N	Väisälä-Brunt (buoyancy) frequency	23	1
p	Pressure perturbation	41	10
P	Undisturbed pressure	37	3
P_o	General atmospheric pressure (= P + p)	41	8
r	Symbol for $4f^4/(f^4 - k^2g^2)$	49	43
R_m	Specific gas constant (2.88×10^6 ergs/gram-deg)	36	2
s	Symbol for Mh/f^2	53	—
T	Absolute temperature (at height z)	21	—
T	Period of the wave motion ($2\pi/\sigma$)	47	—
u	x-component of the velocity perturbation $\vec{v}$	44	22
$\bar{u}$	Mean wind speed in direction of wave propagation	57	—
U	x-component of the undisturbed wind velocity $\vec{V}$	44	22
v	y-component of the velocity perturbation $\vec{v}$	44	22
$\vec{v}$	Velocity perturbation	38	7
v_x, v_y, v_z	Scalar components of the velocity perturbation $\vec{v}$	38	7
V	y-component of the undisturbed wind velocity $\vec{V}$	44	22
$\vec{V}$	Undisturbed horizontal wind velocity	37	4
V_x, V_y	Scalar components of the undisturbed velocity $\vec{V}$	37	4
$\vec{V}_o$	General atmospheric velocity ($= \vec{V} + \vec{v}$)	41	8
w	z-component of the velocity perturbation $\vec{v}$	44	22
x,y	Usual rectangular coordinates (horizontal)	47	34
z	Vertical coordinate (positive upward)	21	—

Greek Letters

β	Environmental lapse rate of dry air	21	—
β_{ad}	Dry adiabatic lapse rate (10 deg/km)	21	—
γ	Ratio of specific heats at constant pressure and constant volume for dry air (= 1.4)	37	5
δ	Symbol for $h + f^2k_z/M$	52	54
ϵ	Symbol for $h - f^2k_z/M$	52	54
θ	Layer width $z_n - z_{n-1}$	53	—
λ	Wavelength ($2\pi/k$)	57	—
ρ	Density perturbation	41	10
ϱ	Undisturbed density (at height z)	37	3
ρ_o	General atmospheric density ($= \varrho + \rho$)	41	8

Symbol	Explanation	Page	Equation
σ	Angular frequency of the wave motion (2π/period)	47	34
Σ	Total gravitational potential	41	8
χ	Divergence of perturbation velocity $(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})$	45	27
Ω	Angular velocity of the earth	41	8
∇	Vector differential operator	41	—
$(')$	Denotes differentiation with respect to z	45	—

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