UNSTEADY FLOW IN CONTIGUOUS AQUIFERS OF DIFFERENT HYDRAULIC PROPERTIES

by

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Abstract

Non-steady potential distribution is found for two flow systems. Both systems are of uniform thickness and are contained in a single horizontal layer. The layer is composed of two homogeneous aquifers bounded above and below by impermeable boundaries. The first system considered is "flow toward a slit" (plane sink). Infinite series solutions are presented for two cases (constant drawdown at the slit and constant discharge at the slit).

The series converge rapidly for small values of time; however, for larger values of time an approximate solution is developed. Special cases are developed from the general solutions by letting the formation constants assume specific values. Graphs, showing the behavior of the variation of drawdown with time, are constructed.

The second system considered is that of "flow toward a well in an infinite strip of constant width". Three cases are considered: (A) An infinite strip having zero drawdown maintained on both boundaries; (B) An infinite strip with one boundary maintained at zero drawdown and the other at zero flux; (C) An infinite strip with both boundaries maintained at zero flux. Graphs, showing the behavior of the steady-state solutions, using several values for the ratio of the hydraulic conductivities, are constructed.

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PART I

INTRODUCTION AND STATEMENT OF PROBLEMS

INTRODUCTION

Mathematical models have been used successfully in solving many flow problems in the quantitative analysis of ground-water resources. Applications of these solutions depend on the assumption that the water-bearing strata are entirely homogeneous in character; most strata, however, are more or less heterogeneous. If the hydraulic properties of the material do not vary greatly, the medium may be considered to have an average homogeneity that is proportional to its average hydraulic properties. These average properties can be obtained from several individual measurements, thus making it advantageous to use mathematical solutions.

Commonly, water-bearing media consist of alternate layers and/or contiguous deposits of markedly different character. Although such layer or contiguous deposit has its own average individual homogeneity, to assume an average homogeneity for the whole aquifer is undoubtedly erroneous.

Mathematical models for steady flow of fluids in such systems have been devised and solved by several authorities (e.g., Kirkham, 1951, 1954; Kochina, 1938, 1939, 1940, 1941, 1942; Muskat, 1946; Hantush and Jacob, 1954, 1955, 1956; Hantush, 1957, 1959). However, solutions for nonsteady flow problems in such systems are not readily available.

PURPOSE

The purpose of this study is to set up mathematical models and obtain solutions for some nonsteady-flow problems in contiguous aquifers of different properties. Several problems of practical interest require the determination of the drawdown variation with time and distance produced by the flow toward a region of discharge. Such a region drains an aquifer on whose boundaries the head and/or flux are known. Although solutions satisfying all possible field conditions are beyond the scope of this study, some general solutions and, when possible, the corresponding steady-state solutions are presented.

STATEMENT OF THE PROBLEMS

The problems treated in the present study are: (1) The flow toward a slit. Two cases are presented. In case A, the drawdown at the slit is constant; in case B, the discharge at the slit is constant. (2) The flow toward a steadily discharging well in an infinite strip. Three cases are presented. Case A is that of zero drawdown on both boundaries of the strip; case B is that of zero drawdown on one boundary and zero flux across the other; case C is that of zero flux across both boundaries.

(1) Flow toward a slit

This system may simulate the flow toward an infinitely long slit parallel to the surface of contact between two aquifers in a single water-bearing layer. Each aquifer has different hydraulic properties. The slit is sufficiently distant from the contact, and the contact surface is considered normal to the layering. Such an assumption may be valid if the layering is practically horizontal. In considering the flow in such a system, two cases of practical interest may be encountered (see Fig. 1). These are:

Case A. Constant drawdown at the slit

In practice, this system may correspond to the flow toward a line of closely spaced flowing wells or natural springs, or toward a stream that cuts through an aquifer. The head along the line of discharge is uniform. Aquifers draining into the sea may fall under such a system of flow; likewise, with certain restrictions, a deep open drain whose hydraulic head is maintained constant.

Case B. Constant discharge from the slit

This system may simulate the flow toward a line of wells or centers of pumping, closely spaced and pumping at a reasonably constant discharge. With certain restrictions, it also corresponds to the flow toward a deep open drain whose discharge is removed at a constant rate.

(2) Flow toward a steadily discharging well in an infinite strip

Consider a well located in an aquifer. The aquifer is in the form of an infinite strip on whose boundaries the drawdown and/or flux is zero. The two infinite parallel boundaries are perpendicular to the plane of contact between two facies of a water-bearing layer (Fig. 2). Three cases, in which either the drawdown, the flux, or both vanish on the boundaries, arise in considering the flow to a steadily discharging well in such a system. These are:

Case A. Zero drawdown on both boundaries

This system corresponds to the flow to a well located between two parallel rivers or canals. The streams may simulate straight-line boundaries along which the head is essentially uniform; it is assumed that they cut completely through the aquifer and are of large capacity and have small slopes. The flow toward a well between a fairly long lake and a river or canal parallel to the lake may fall under such a system.

Case B. Zero drawdown on one boundary and zero flux across the other

A well draining an aquifer that is closed on one side and open to a fairly long stream or to the sea on the other side may fall under this system of flow. The stream and the impermeable boundary (fault, bedrock, wall, or ledge) are parallel to each other.

Case C. Zero flux across both boundaries

This system may simulate the flow toward a well draining an aquifer bounded by two parallel faults, or bedrock walls, or ledges. It may represent the flow, with certain limitations, in a fairly long body of perched water.

Although the solutions obtained are necessarily for artesian conditions (confined flow), they can still be used, with limitations, if the flow is under water-table conditions (unconfined flow).

ASSUMPTIONS

In addition to the usual assumptions concerning elasticity, uniformity, homogeneity of each aquifer, and constancy of the formation constants in space and time, the following additional assumptions are made: (1) The flow in the regions considered does not vary with depth, so that the three-dimensional problem becomes a problem of one- or two-dimensional flow, depending upon the conditions of the problem. (2) The lateral boundaries of the regions considered are vertical and cut completely through the artesian aquifer. The conditions along these boundaries are uniform in the vertical direction.

DIFFERENTIAL EQUATION OF GROUND-WATER MOTION

The differential equation (Muskat, 1946; Jacob, 1946; Hantush and Jacob, 1954, 1955, 1956; etc.) for nonsteady flow of ground water is:

$$\nabla^2 s = \frac{1}{\nu} \frac{\partial s}{\partial t}$$

where s is the drawdown at any point (x, y) in the region and at any time t since pumping started; S is the storage coefficient (volume of water that a unit decline of head releases from storage in a vertical prism of aquifer of unit cross-section); T = Kb is the transmissibility of the aquifer (discharge per unit normal width per unit of decline of hydraulic head), $\nu = T/S$, and ∇^2 is the Laplacian operator.

The solutions of this equation that apply to any particular problem of interest depend upon the detailed physical conditions imposed at the boundaries of the fluid system and on the initial distribution of the drawdown. In the steady state, the equation reduces to ($\nabla^2 s = 0$), the well-known Laplace equation, which frequently occurs in other branches of physics.

The procedure in obtaining solutions is the following: Elementary solutions of the partial differential equation are obtained separately for each region and adjusted to fit the boundary conditions. The method of the Laplace transformation in solving differential equations is used in solving the problems under consideration. At the surface of contact separating the two regions, the conditions which must be satisfied are: The drawdown across it, and the normal velocity must be continuous.

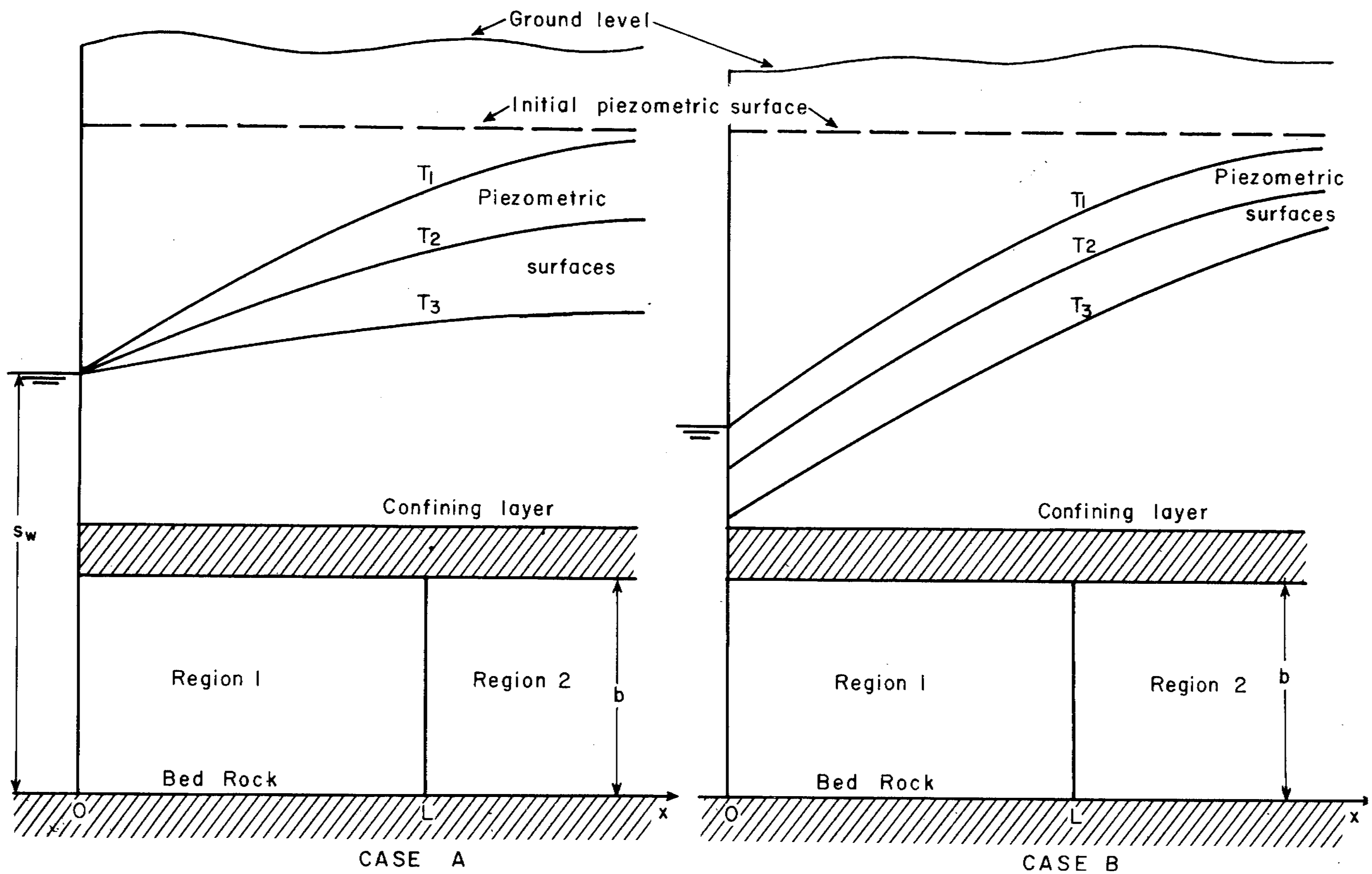


Fig. 1 -- Diagramatic representation of flow toward a slit draining an aquifer composed of two contiguous regions.

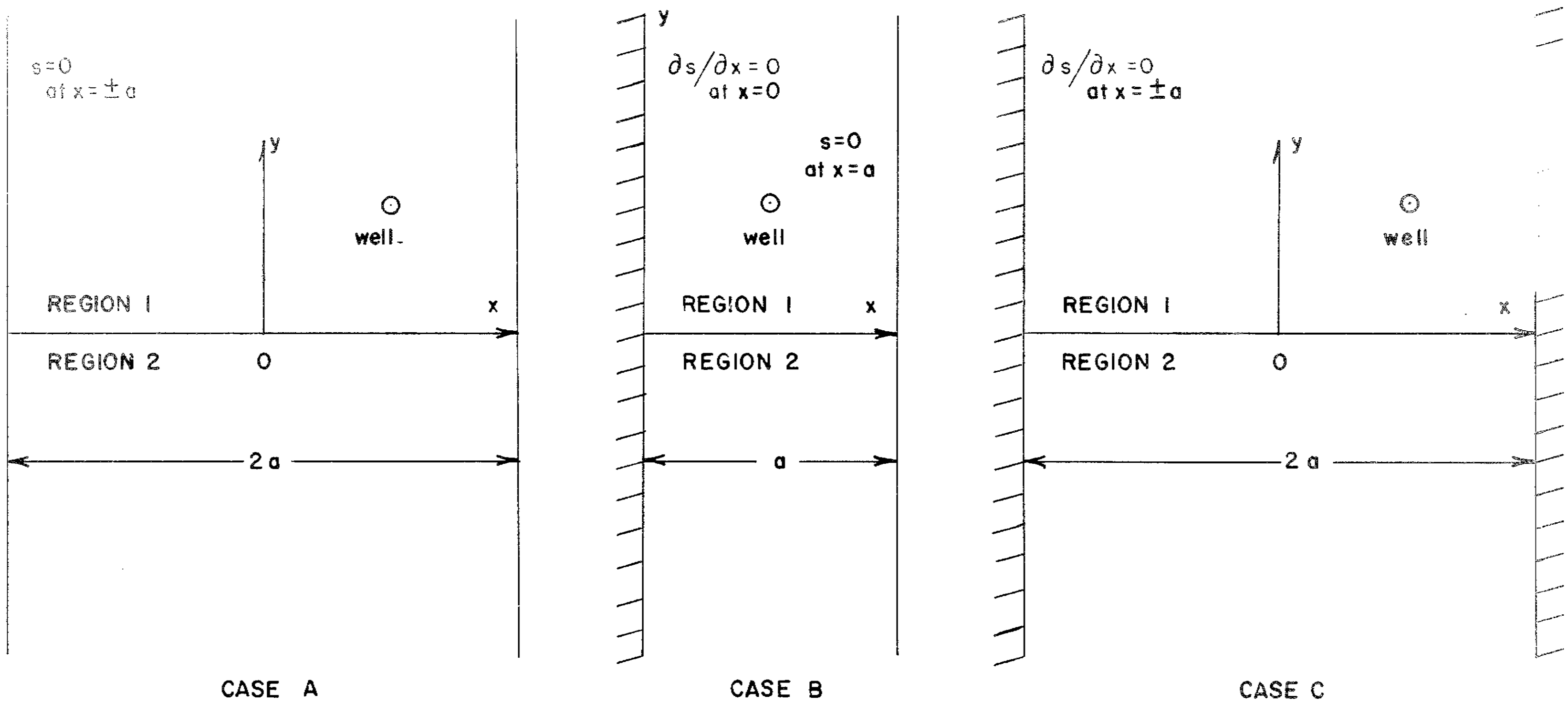


Fig. 2 -- Diagramatic representation of two-dimensional flow toward a well in an infinite strip of aquifer composed of two contiguous regions.

P A R T   I I

A N A L Y S E S

FLOW SYSTEMS

PROBLEM 1. FLOW TOWARD A SLIT

Case A. Constant Drawdown at the Slit

The problem is to determine the variation with time of the drawdown induced by a slit (plane sink) maintained at a constant drawdown and draining an artesian aquifer of two contiguous regions of different hydraulic properties. The flow system of this case is represented diagramatically by Figure 1.

The slit can be considered, for analytical purposes, as a plane sink placed parallel to, and at a distance L from, the interface between the two contiguous regions. The required solution will be that of the following system of equations:

$$\left.\begin{aligned} &\frac{\partial^2 s_1}{\partial x^2} = \frac{1}{\nu_1}\frac{\partial s_1}{\partial t} \;\ldots\ldots\ldots\ldots\; \text{(a)} \\ &s_1(x,0) = 0, \quad 0 \leq x \leq L \;\ldots\; \text{(b)} \\ &s_1(0,t) = s_w, \quad t \geq 0 \;\ldots\ldots\; \text{(c)} \end{aligned}\right| \;\ldots\ldots\ldots\ldots\; (1)$$

$$\frac{\partial^2 s_2}{\partial x^2} = \frac{1}{\nu_2}\frac{\partial s_2}{\partial t} \quad \ldots\ldots\ldots \text{(a)}$$

$$s_2(x,0) = 0, \quad x \geq L \quad \ldots \text{(b)} \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

$$s_2(\infty,t) = 0, \quad t \geq 0 \quad \ldots \text{(c)}$$

$$s_1(L,t) = s_2(L,t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{\partial s_1(L,t)}{\partial x} = \delta \frac{\partial s_2(L,t)}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where subscripts 1 and 2 refer to regions 1 and 2, respectively; and $\delta = K_2/K_1 = T_2/T_1$ is the ratio of the transmissibilities of the two contiguous regions. The other terms are as defined before.

Solution of the Problem -- The Laplace transforms of the above equations with respect to time, under conditions (1b) and (2b), are:

$$\frac{\partial^2 \bar{s}_1}{\partial x^2} = \frac{p}{\nu_1}\bar{s}_1 \quad \ldots\ldots\ldots\ldots\ldots \text{(a)}$$

$$\bar{s}_1(0,p) = s_w/p, \quad p \geq 0 \quad \ldots \text{(b)} \qquad \ldots\ldots\ldots\ldots (5)$$

$$\frac{\partial^2 \bar{s}_2}{\partial x^2} = \frac{p}{\nu_2} \bar{s}_2 \quad \text{.................} \quad (a)$$

$$\bar{s}_2(\infty, p) = 0 \quad \text{.................} \quad (b)$$

(6)

$$\bar{s}_1(L, p) = \bar{s}_2(L, p), \; p \geqq 0 \quad \text{......................} \quad (7)$$

$$\frac{\partial \bar{s}_1(L, p)}{\partial x} = \delta \frac{\partial \bar{s}_2(L, p)}{\partial x} \quad \text{......................} \quad (8)$$

where $\bar{s}_1$ and $\bar{s}_2$ are the Laplace transforms with respect to time of s_1 and s_2, and p is the parameter obtained by using the Laplace transformation.

Equations (5a) and (6a) are satisfied by the following solutions:

$$\bar{s}_1 = c_1 \cosh\left(x\sqrt{p/\nu_1}\right) + c_2 \sinh\left(x\sqrt{p/\nu_1}\right) \quad \ldots \quad (9)$$

$$\bar{s}_2 = c_3 \exp\left(x\sqrt{p/\nu_2}\right) + c_4 \exp\left(-x\sqrt{p/\nu_2}\right) \quad \ldots \quad (10)$$

in which c_1, c_2, c_3, and c_4 have the following values obtained by using the remaining boundary conditions:

$$c_1 = \frac{s_w}{p}$$

$$c_2 = -\frac{s_w}{p}\left\{\frac{\sqrt{p/\nu_1}\,\sinh\left(L\sqrt{p/\nu_1}\right) + \delta\sqrt{p/\nu_2}\,\cosh\left(L\sqrt{p/\nu_1}\right)}{\sqrt{p/\nu_1}\,\cosh\left(L\sqrt{p/\nu_1}\right) + \delta\sqrt{p/\nu_2}\,\sinh\left(L\sqrt{p/\nu_1}\right)}\right\}$$

$$c_3 = 0$$

$$c_4 = \frac{s_w}{p} \left\{ \frac{\sqrt{p/\nu_1} \exp\left(L \sqrt{p/\nu_2}\right)}{\sqrt{p/\nu_1} \cosh\left(L \sqrt{p/\nu_1}\right) + \delta \sqrt{p/\nu_2} \sinh\left(L \sqrt{p/\nu_1}\right)} \right\}$$

Substituting these constants in equations (9) and (10) and then simplifying, one obtains:

$$\bar{s}_1 = \frac{s_w}{p} \left\{ \frac{\sqrt{p/\nu_1} \cosh\left[(L-x)\sqrt{p/\nu_1}\right] + \delta \sqrt{p/\nu_2} \sinh(L-x)\sqrt{p/\nu_1}}{\sqrt{p/\nu_1} \cosh\left(L \sqrt{p/\nu_1}\right) + \delta \sqrt{p/\nu_2} \sinh\left(L \sqrt{p/\nu_1}\right)} \right\}. \quad (11)$$

$$\bar{s}_2 = \frac{s_w}{p} \left\{ \frac{\sqrt{p/\nu_1} \exp\left[(L-x)\sqrt{p/\nu_2}\right]}{\sqrt{p/\nu_1} \cosh\left(L \sqrt{p/\nu_1}\right) + \delta \sqrt{p/\nu_2} \sinh\left(L \sqrt{p/\nu_1}\right)} \right\}. \quad (12)$$

If the hyperbolic functions in (11) and (12) are replaced by their exponential forms, inasmuch as $1/(1+z) = \sum_{n=0}^{\infty} (-)^n z^n$, if $|z| < 1$, $\bar{s}_1$ and $\bar{s}_2$ may finally be written as:

$$\bar{s}_1 = \frac{s_w}{p} \left\{ \sum_{n=0}^{\infty} (-)^n \gamma^n e^{-(x+2nL)\sqrt{p/\nu_1}} + \gamma \sum_{n=0}^{\infty} (-)^n \gamma^n e^{-[2L(n+1)-x]\sqrt{p/\nu_1}} \right\} \quad \ldots\ldots\ldots\ldots \quad (13)$$

$$\bar{s}_2 = \frac{s_w}{p} (1+\gamma) \sum_{n=0}^{\infty} (-)^n \gamma^n \exp\left(-\left[\frac{(x-L)}{\sqrt{\nu_2}} + \frac{L(2n+1)}{\sqrt{\nu_1}}\right]\sqrt{p}\right) \ldots \quad (14)$$

where $\gamma = \dfrac{\sqrt{\nu_2} - \delta\sqrt{\nu_1}}{\sqrt{\nu_2} + \delta\sqrt{\nu_1}}$.

It is easily seen that equations (13) and (14) satisfy conditions (5b), (6b), (7), and (8).

The inverse transforms of equations (13) and (14) are obtainable from tables of Laplace transforms [Churchill, 1938, p.328]. The inverse transformations of (13) and (14) can be reduced finally to:

$$s_1 = s_w \left\{ \sum_{n=0}^{\infty} (-)^n \gamma^n \operatorname{erfc}\left(\frac{x+2nL}{\sqrt{4\nu_1 t}}\right) \right.$$

$$\left. + \gamma \sum_{n=0}^{\infty} (-)^n \gamma^n \operatorname{erfc}\left(\frac{2L(n+1)-x}{\sqrt{4\nu_1 t}}\right) \right\} \ldots\ldots\ldots\ldots\ldots \quad (15)$$

$$s_2 = s_w (1+\gamma) \sum_{n=0}^{\infty} (-)^n \gamma^n \operatorname{erfc}\left\{\frac{(x-L)}{\sqrt{4\nu_2 t}} + \frac{L(2n+1)}{\sqrt{4\nu_1 t}}\right\} \ldots \quad (16)$$

where $\operatorname{erfc}(Z) = \int_Z^{\infty} e^{-\beta^2} d\beta$.

Equations (15) and (16) describe the drawdown distribution in the flow system. The series in (15) and (16) are rapidly convergent for small values of time and lend themselves to relatively easy calculation.

<u>Solution for Large Values of Time</u> -- For relatively small values of time, the infinite series in (15) and (16) converge fairly rapidly; however, for larger values of time ($t \geq 100\ L^2/\nu_1$), an approximate solution can be obtained as follows: For $(L-x)\sqrt{p/\nu_1} \leq .1$ (or $t \geq 100\ L^2/\nu_1$), $\cosh[(L-x)\sqrt{p/\nu_1}] \cong 1$, and $\sinh[(L-x)\sqrt{p/\nu_1}] \cong (L-x)\sqrt{p/\nu_1}$.

Using these approximations in equations (11) and (12), one obtains:

$$\bar{s}_1 = \frac{s_w}{p}\left\{\frac{1 + \delta\,(L-x)\sqrt{p/\nu_2}}{1 + \delta L\sqrt{p/\nu_2}}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

$$\bar{s}_2 = \frac{s_w}{p}\left\{\frac{\exp[-(x-L)\sqrt{p/\nu_2}]}{1 + \delta L\sqrt{p/\nu_2}}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

The inverse transforms of (17) and (18) are easily found to be [Churchill, 1958, p.326]:

$$s_1 = s_w \left\{ 1 - (x/L) \exp\left(\frac{\nu_2 t}{\delta^2 L^2}\right) \operatorname{erfc}\left(\frac{\sqrt{\nu_2 t}}{\delta L}\right) \right\} \quad \ldots\ldots \quad (19)$$

$$s_2 = s_w \left\{ \operatorname{erfc}\left(\frac{x-L}{\sqrt{4\nu_2 t}}\right) - \exp\left(\frac{x-L}{\delta L}\right) \exp\left(\frac{\nu_2 t}{\delta^2 L^2}\right) \operatorname{erfc}\left[\frac{\sqrt{\nu_2 t}}{\delta L} + \frac{x-L}{\sqrt{4\nu_2 t}} \right] \right. \quad . \quad (20)$$

Approximate Solution at the Interface -- At the interface of the two regions, $s_L = s_1 = s_2$ and $x = L$. With these substitutions, either (19) or (20) becomes:

$$s_L = s_w \left\{ 1 - \exp\left(\frac{\nu_2 t}{\delta^2 L^2}\right) \operatorname{erfc}\left(\frac{\sqrt{\nu_2 t}}{\delta L}\right) \right\} \quad \ldots\ldots\ldots\ldots \quad (21)$$

Discharge of the Slit -- The discharge of the slit can be obtained by using Darcy's law in the form:

$$-T_1 \frac{\partial}{\partial x} s_1(0,t) = q$$

where q is the flow into the slit per unit of thickness of aquifer.

Using equation (15), one obtains:

$$q = \frac{s_w T_1}{\sqrt{\pi \nu_1 t}} \{ \sum_{n=0}^{\infty} (-)^n \gamma^n \exp[-n^2 L^2 / \nu_1 t]$$

$$- \gamma \sum_{n=0}^{\infty} (-)^n \gamma^n \exp[L^2 (n+1)^2 / \nu_1 t] \} \quad \ldots\ldots\ldots\ldots \quad (22)$$

An approximate solution for the discharge of the slit can be obtained by using equation (19) and Darcy's law:

$$q = \frac{s_w T_1}{L} \exp(\nu_2 t / \delta^2 L^2) \ \text{erfc} \ (\frac{\sqrt{\nu_2 t}}{\delta L}) \quad \ldots\ldots\ldots\ldots \quad (23)$$

where q is the flow per unit thickness of aquifer for $x = 0$.

<u>Solutions for Special Cases</u> -- Three solutions of practical interest occur when the formation constants in equations (15) and (16) assume specific values. These are: (a) A homogeneous region extending indefinitely from the slit; (b) a homogeneous region bounded by the slit and by an impermeable boundary parallel to the plane of the slit; and (c) a homogeneous region bounded by the slit and a region of infinite conductivity, such as a river or a lake of fairly straight and long shores.

Special Case (a) -- Drawdown distribution in a homogeneous region of large extent drained by a slit at constant drawdown. When the hydraulic constants in regions 1 and 2 are identical and L is set equal to zero, equation (16) will reduce to the required expression:

$$s = s_w \operatorname{erfc} \sqrt{u_x} \quad \ldots\ldots \quad (24)$$

where

$$u_x = x^2/4\nu t.$$

Special Case (b) -- Drawdown distribution in a region between a slit and an impermeable boundary. If the hydraulic conductivity of region 2 is set equal to zero, this region represents an impermeable boundary parallel to, and at distance L from, the plane of the slit. As K_2 approaches zero, ν_2 approaches zero also. Thus, if ν_2 goes to zero, $\gamma = (\sqrt{\nu_2} - \delta\sqrt{\nu_1})/(\sqrt{\nu_2} + \delta\sqrt{\nu_1})$ approaches the limiting value of (-1). With these substitutions, equation (15) becomes:

$$s = s_w \sum_{n=0}^{\infty} \{ \operatorname{erfc} \sqrt{u_{n1}} - \operatorname{erfc} \sqrt{u_{n2}} \} \quad \ldots\ldots \quad (25)$$

where

$$u_{n1} = (x+2nL)^2/4\nu_1 t \quad \text{and} \quad u_{n2} = [2L(n+1)-x]^2/4\nu_1 t.$$

Special Case (c) -- Drawdown distribution in an aquifer bounded by a source of recharge of very high hydraulic conductivity and drained by a slit parallel to that boundary. This solution can be obtained by letting the conductivity, and hence the transmissibility, in region 2 become infinite. In the limit, as K_2 approaches infinity, γ approached (+1). The limit is obtained by using L'hopital's rule of limits. Making these substitutions in equation (15), one obtains:

$$s = s_W \sum_{n=0}^{\infty} (-)^n \{\text{erfc} \sqrt{u_{n1}} + \text{erfc} \sqrt{u_{n2}}\} \quad \ldots\ldots (26)$$

The series in equations (25) and (26) converge fairly rapidly and are suitable for calculation.

Case B. Constant Discharge at the Slit

This problem is the same as that of case A, except that the slit, instead of being maintained at a constant head, is discharging at a constant rate. The required solution is that of the following system of equations:

$$\frac{\partial^2 s_1}{\partial x^2} = \frac{1}{\nu_1}\frac{\partial s_1}{\partial t} \quad \text{(a)}$$

$$s_1(x,0) = 0, \quad 0 \le x \le L \quad \text{(b)} \qquad \text{(27)}$$

$$q = -T_1 \frac{\partial}{\partial x} s_1(0,t) \quad \text{(c)}$$

$$\frac{\partial^2 s_2}{\partial x^2} = \frac{1}{\nu_2}\frac{\partial s_2}{\partial t} \quad \text{(a)}$$

$$s_2(x,0) = 0, \quad x \ge L \quad \text{(b)} \qquad \text{(28)}$$

$$s_2(\infty,t) = 0, \quad t \ge 0 \quad \text{(c)}$$

$$\frac{\partial s_1(L,t)}{\partial x} = \delta \frac{\partial s_2(L,t)}{\partial x} \qquad \text{(29)}$$

$$s_1(L,t) = s_2(L,t) \qquad \text{(30)}$$

where q represents the constant discharge at the face of the slit.

Solution of the Problem --- The Laplace transforms of the above equations with respect to time, under conditions (27b) and (28b), are:

$$\frac{\partial^2 \bar{s}_1}{\partial x^2} = \frac{p}{\nu_1}\bar{s}_1 \quad \text{(a)} \qquad \text{(31)}$$

$$\frac{\partial \bar{s}_1}{\partial x} = \frac{-q}{T_1 p} \quad \text{(b)}$$

$$\left.\begin{aligned} &\frac{\partial^2 \bar{s}_2}{\partial x^2} = \frac{p}{\nu_2}\bar{s}_2 \ \ldots\ldots\ldots\ldots \text{(a)} \\ &\bar{s}_2(\infty,t) = 0 \ \ldots\ldots\ldots\ldots \text{(b)} \end{aligned}\right| \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (32)$$

$$\frac{\partial \bar{s}_1(L,p)}{\partial x} = \delta \frac{\partial \bar{s}_2(L,p)}{\partial x} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (33)$$

$$\bar{s}_1(L,p) = \bar{s}_2(L,p) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (34)$$

It can be shown that the solutions satisfying conditions (31a) and (32a) are given respectively by:

$$\bar{s}_1 = D_1 \cosh(x\sqrt{p/\nu_1}) + D_2 \sinh(x\sqrt{p/\nu_2}) \ \ldots\ldots \ (35)$$

$$\bar{s}_2 = D_3 \exp(-x\sqrt{p/\nu_2}) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (36)$$

By using the remaining conditions, the arbitrary constants in (35) and (36) are:

$$D_1 = \frac{-q}{T_1 p\sqrt{p/\nu_1}}$$

$$\left\{ \frac{\sqrt{p/\nu_1}\cosh(L\sqrt{p/\nu_1}) + \delta\sqrt{p/\nu_2}\sinh(L\sqrt{p/\nu_1})}{\sqrt{p/\nu_1}\sinh(L\sqrt{p/\nu_1}) + \delta\sqrt{p/\nu_2}\cosh(L\sqrt{p/\nu_1})} \right\}$$

$$D_2 = -q \frac{\sqrt{\nu_1/p}}{T_1 p}$$

$$D_3 = \frac{-q}{T_1 p \sqrt{p/\nu_1}} \exp(L\sqrt{p/\nu_2}) \left[\sinh(L\sqrt{p/\nu_1}) \right.$$

$$- \left\{ \frac{\sqrt{p/\nu_1}\cosh(L\sqrt{p/\nu_1}) + \delta\sqrt{p/\nu_2}\sinh(L\sqrt{p/\nu_1})}{\sqrt{p/\nu_1}\sinh(L\sqrt{p/\nu_1}) + \delta\sqrt{p/\nu_2}\cosh(L\sqrt{p/\nu_1})} \right\}$$

$$\left. \cosh(L\sqrt{p/\nu_1}) \right]$$

Upon replacing these constants in equations (35) and (36) and simplifying, they become:

$$\bar{s}_1 = \frac{-q\sqrt{\nu_1}}{T_1 p^{3/2}}$$

$$\left\{ \frac{\delta\sqrt{p/\nu_2}\sinh[(x-L)\sqrt{p/\nu_1}] - \cosh[(x-L)\sqrt{p/\nu_1}]}{\sqrt{p/\nu_1}\sinh(L\sqrt{p/\nu_1}) + \delta\sqrt{p/\nu_2}\cosh(L\sqrt{p/\nu_1})} \right\} \quad \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (37)$$

$$\bar{s}_2 = \frac{-q}{T_1 p} \left\{ \frac{\exp[-(x-L)\sqrt{p/\nu_2}\,]}{\sqrt{p/\nu_1}\,\sinh(L\sqrt{p/\nu_1}\,) + \delta\sqrt{p/\nu_2}\,\cosh(L\sqrt{p/\nu_1}\,)} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (38)$$

Replacing the hyperbolic functions in (37) and (38) by their exponential forms, and recalling that $1/(1-z) = \sum_{n=0}^{\infty} z^n$, if $|z| < 1$, one obtains after simplifying:

$$\bar{s}_1 = \frac{q\sqrt{\nu_1}}{T_1 p^{3/2}} \left\{ \gamma \sum_{n=0}^{\infty} \gamma^n \exp\{[\,x-2L(n+1)\,]\sqrt{p/\nu_1}\,\} + \sum_{n=0}^{\infty} \gamma^n \exp[-(x+2nL)\sqrt{p/\nu_1}\,] \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (39)$$

$$\bar{s}_2 = \frac{q\sqrt{\nu_1}\,(1+\gamma)}{T_1 p^{3/2}} \sum_{n=0}^{\infty} \gamma^n \exp\left\{-\left[\frac{(x-L)\sqrt{\nu_1} + (2n+1)L\sqrt{\nu_2}}{\sqrt{\nu_1\nu_2}}\right]\sqrt{p}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

After obtaining the inverse transforms [Churchill 1958, p. 328] of (39) and (40), the drawdown equations will be given, after simplification, by:

$$s_1 = \frac{q\sqrt{\nu_1}}{T_1}\Big[\, 2\bar{\gamma}\sqrt{t/\pi}\sum_{n=0}^{\infty}\bar{\gamma}^{n}\exp\Big\{-\frac{[2L(n+1)-x]^2}{4\nu_1 t}\Big\}$$

$$+\,2\sqrt{t/\pi}\sum_{n=0}\bar{\gamma}^{n}\exp\Big\{-\frac{[x+2nL]^2}{4\nu_1 t}\Big\}$$

$$-\,\bar{\gamma}\sum_{n=0}^{\infty}\bar{\gamma}^{n}\frac{[2L(n+1)-x]}{\sqrt{\nu_1}}\operatorname{erfc}\Big\{\frac{2L(n+1)-x}{\sqrt{4\nu_1 t}}\Big\}$$

$$-\sum_{n=0}^{\infty}\gamma^{n}\frac{[x+2nL]}{\sqrt{\nu_1}}\operatorname{erfc}\Big\{\frac{x+2nL}{\sqrt{4\nu_1 t}}\Big\}\Big] \quad \ldots\ldots\ldots\ldots\ldots \quad (41)$$

$$s_2 = \frac{q\sqrt{\nu_1}\,(1+\gamma)}{T_1}$$

$$\Big[\, 2\sqrt{t/\pi}\ \sum_{n=0}^{\infty} \gamma^n \exp\{ -\frac{[(x-L)\sqrt{\nu_1} + (2n+1)L\sqrt{\nu_2}\,]^2}{4t\nu_1\nu_2} \}$$

$$- \sum_{n=0}^{\infty} \gamma^n \Big[\frac{(x-L)\sqrt{\nu_1} + (2n+1)L\sqrt{\nu_2}}{\sqrt{\nu_1\nu_2}} \Big]$$

$$\operatorname{erfc}\Big[\frac{(x-L)\sqrt{\nu_1} + (2n+1)L\sqrt{\nu_2}}{\sqrt{4t\nu_1\nu_2}} \Big] \quad \ldots\ldots\ldots\ldots\ldots \quad (42)$$

Equations (41) and (42) completely describe the drawdown distribution in the system. It is easily seen that they satisfy the boundary conditions (27), (28), (29), and (30). In order to find the derivative of the complementary error function, the generalized Leibnitz formula [Churchill, 1958, p.30] is applied to the function in its integral form. The series of (41) and (42) are rapidly convergent for small values of time. Only a few terms are required to obtain results accurate to four decimal places.

Solution for Large Values of Time -- For larger values of time, an approximate solution can be obtained by a process similar to that used under case A. This solution (provided $t \geq 100\ L^2/\nu_1$) is:

$$s_1 = \frac{q}{T_1}\left\{ \frac{L\alpha_2}{\delta^2\alpha_1}\left[\left(\frac{\delta\alpha_1}{\sqrt{\alpha_2}}\frac{2}{\sqrt{\pi}} - 1\right) + \exp\left(\frac{\delta^2\alpha_1^2}{\alpha_2}\right)\operatorname{erfc}\left(\frac{\delta\alpha_1}{\sqrt{\alpha_2}}\right)\right]\right.$$

$$\left. + (x/L-1)L\left[\exp\left(\frac{\delta^2\alpha_1^2}{\alpha_2}\right)\operatorname{erfc}\left(\frac{\delta\alpha_1}{\sqrt{\alpha_2}}\right) - 1\right]\right\} \quad \ldots\ldots\ (43)$$

$$s_2 = \frac{q}{T_1}\left\{ \frac{L\alpha_2}{\delta^2\alpha_1}\left[\left(\frac{\delta\alpha_1}{\sqrt{\alpha_2}}\frac{2}{\sqrt{\pi}}\exp\left\{-\frac{[x/L-1]^2}{4\alpha_2}\right\} - \operatorname{erfc}\left\{\frac{[x/L-1]}{\sqrt{4\alpha_2}}\right\}\right.\right.\right.$$

$$\left.\left. + \exp\left\{\frac{\delta\alpha_1}{\alpha_2}(x/L-1)\right\}\exp\left\{\frac{\delta^2\alpha_1^2}{\alpha_2}\right\}\operatorname{erfc}\left\{\frac{\delta\alpha_1}{\sqrt{\alpha_2}} + \frac{(x/L-1)}{\sqrt{4\alpha_2}}\right\}\right]\right]$$

$$\left. - \frac{(x/L-1)}{\delta}\operatorname{erfc}\left\{\frac{(x/L-1)}{\sqrt{4\alpha_2}}\right\}\right. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (44)$$

where $\alpha_1 = \nu_1 t/L^2$, and $\alpha_2 = \nu_2 t/L^2$. The other symbols are as defined before.

Equations (43) and (44) may be simplified further by using the series representation of the complementary error function:

$$\frac{\sqrt{\pi}}{2}\,\mathrm{erfc}(z) = \frac{1}{2}\, e^{-z^2}\left[\frac{1}{z} + \sum_{r=2}^{\infty} (-)^{r-1} \frac{1\cdot 3 \ldots (2r-3)}{2^{r-1}\, z^{2r-1}}\right]$$

For $z \geq 10$, little error is introduced by neglecting all of the terms of the series. Thus, for $t \geq \frac{121 L^2 \nu_2}{\delta^2 \nu_1^2}$ or $t \geq \frac{100 L^2}{\nu_1}$, whichever is greater, equations (43) and (44), after making this approximation, will become:

$$s_1 = \frac{q}{T_1}\left\{ \frac{L\alpha_2}{\delta^2 \alpha_1}\left[\frac{2}{\sqrt{\pi}}\frac{\delta\alpha_1}{\sqrt{\alpha_2}} - 1\right] + \frac{1}{\sqrt{\pi}}\frac{\sqrt{\alpha_2}}{\delta\alpha_1} \right.$$

$$\left. + (x/L-1)L\left[\frac{1}{\sqrt{\pi}}\frac{\sqrt{\alpha_2}}{\delta\alpha_1} - 1\right]\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (45)$$

$$s_2 = \frac{q}{T_1} \left\{ \frac{L\alpha_2}{\delta^2 \alpha_1} \left[\frac{\delta\alpha_1}{\sqrt{\alpha_2}} \frac{2}{\sqrt{\pi}} \exp\left\{ \frac{-(x/L-1)^2}{4\alpha_2} \right\} - \operatorname{erfc}\left\{ \frac{(x/L-1)}{\sqrt{4\alpha_2}} \right\} \right.\right.$$

$$+ \frac{1}{\sqrt{\pi}} \frac{2\sqrt{\alpha_2}}{2\delta\alpha_1 + (x/L-1)} \exp\left\{ -\frac{1}{4} \frac{(x/L-1)}{\alpha_2} \right\} \Bigg]$$

$$\left. - \frac{(x/L-1)L}{\delta} \operatorname{erfc}\left\{ \frac{(x/L-1)}{\sqrt{4\alpha_2}} \right\} \right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

Solutions for Special Cases -- Three solutions of practical interest can be obtained by a process similar to that used under case A.

Special Case (a) -- Drawdown distribution in a homogeneous region of large extent drained by a constantly discharging slit. When $\nu = \nu_1 = \nu_2$, $T = T_2$ and $L = 0$, equation (42) reduces to the required expression:

$$s = \frac{qx}{T\sqrt{\pi}} \left\{ \frac{e^{-u_x}}{\sqrt{u_x}} - \sqrt{\pi}\, \operatorname{erfc} \sqrt{u_x} \right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

where $u_x = x^2/4\nu t$.

Equation (47) is, as should be, the same as that obtained by Hantush [1955, p. 88]. This equation was used successfully in predicting water levels in the eastern edge of the intake area of the Roswell ground-water reservoir, Roswell, New Mexico.

Special Case (b) -- Drawdown distribution between a slit and an impermeable boundary. When $\nu = \nu_1$, $T = T_1$, and $\gamma = -1$, equation (41) becomes:

$$s = \frac{qx}{T\sqrt{\pi}\sqrt{u_x}} \sum_{n=0}^{\infty} (-)^n \left[\left(e^{-u_{n1}} - e^{-u_{n2}}\right) - \sqrt{\pi}\left(\sqrt{u_{n1}}\,\mathrm{erfc}\sqrt{u_{n1}} - \sqrt{u_{n2}}\,\mathrm{erfc}\sqrt{u_{n2}}\right)\right] \qquad (48)$$

where L is the distance between the slit and the impermeable boundary. The other symbols are as defined before.

Special Case (c) -- Drawdown distribution in an aquifer bounded by a source of recharge of very high hydraulic conductivity and drained by a slit parallel to that boundary. This solution can be obtained by requiring that the transmissibility, and, hence, the conductivity, of region 2 becomes infinitely large. In the limit γ approaches + 1 and equation (41) becomes:

$$s = \frac{qx}{T\sqrt{\pi}\sqrt{u_x}} \sum_{n=0}^{\infty} \left[\left(e^{-u_{n1}} + e^{-u_{n2}} \right) \right.$$

$$\left. - \sqrt{\pi}\left(u_{n1} \operatorname{erfc}\sqrt{u_{n1}} + u_{n2} \operatorname{erfc}\sqrt{u_{n2}} \right) \right] \qquad \ldots\ldots\ldots\ldots (49)$$

where the symbols are as defined before.

PROBLEM 2. FLOW TOWARD A STEADILY DISCHARGING WELL IN AN INFINITE STRIP

Case A. Infinite Strip Composed of Two Contiguous Aquifers Whose Boundaries are Maintained at Zero Drawdown

This nonsteady flow system is produced by a steadily discharging well starting from rest and draining an aquifer in the form of an infinite strip whose boundaries are maintained at constant head. The aquifer is composed of two contiguous regions of different hydraulic properties (see fig. 2, case A). The required solution is that of the following system of equations:

$$\left.\begin{aligned} &\frac{\partial^2 s_1}{\partial x^2} + \frac{\partial^2 s_1}{\partial y^2} = \frac{1}{\nu_1}\frac{\partial s_1}{\partial t} \quad \ldots\ldots \text{(a)} \\ &s_1(\pm a, y, t) = 0,\ t \geq 0 \quad \ldots\ldots \text{(b)} \\ &s_1(x, \infty, t) = 0,\ t \geq 0 \quad \ldots\ldots \text{(c)} \end{aligned}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots (50)$$

$$\left.\begin{array}{ll} \dfrac{\partial^2 s_2}{\partial x^2} + \dfrac{\partial^2 s_2}{\partial y^2} = \dfrac{1}{\nu_2}\dfrac{\partial s_2}{\partial t} & \ldots\ldots \text{(a)} \\ s_2(\pm a, y, t) = 0, \quad t \geq 0 & \ldots\ldots \text{(b)} \\ s_2(x, -\infty, t) = 0, \quad t \geq 0 & \ldots\ldots \text{(c)} \end{array}\right| \qquad (51)$$

$$s_1(x,y,0) = s_2(x,y,0) = 0 \qquad (52)$$

$$\frac{\partial s_1}{\partial y}(x,0,t) = \delta \frac{\partial s_2}{\partial y}(x,0,t) \qquad (53)$$

$$s_1(x,0,t) = s_2(x,0,t) \qquad (54)$$

$$\underset{r \to 0}{\text{Limit}}\; r\frac{\partial s_1}{\partial r} = \frac{-Q}{2\pi T_1} \qquad (55)$$

where 2a is the distance between the parallel boundaries, Q is the discharge of the well, r is the distance from the center of the well, and the well has the coordinates (x_1, y_1). The coordinate system is as shown in Figure 2, case A.

Solution of the problem -- With the use of condition (52), the Laplace transform of the boundary value problem is:

$$\left.\begin{aligned} &\frac{\partial^2 \bar{s}_1}{\partial x^2} + \frac{\partial^2 \bar{s}_1}{\partial y^2} - \frac{p\bar{s}_1}{\nu_1} = 0 \quad \ldots\ldots\ldots\ldots (a) \\ &\bar{s}_1(\pm a, y, p) = 0, \; p \geq 0 \quad \ldots\ldots\ldots\ldots (b) \\ &\bar{s}_1(x, \infty, p) = 0, \; p \geq 0 \quad \ldots\ldots\ldots\ldots (c) \end{aligned}\right| \quad \ldots\ldots\ldots (56)$$

$$\left.\begin{aligned} &\frac{\partial^2 \bar{s}_2}{\partial x^2} + \frac{\partial^2 \bar{s}_2}{\partial y^2} - \frac{p\bar{s}_2}{\nu_2} = 0 \quad \ldots\ldots\ldots\ldots (a) \\ &\bar{s}_2(\pm a, y, p) = 0, \; p \geq 0 \quad \ldots\ldots\ldots\ldots (b) \\ &\bar{s}_2(x, -\infty, p) = 0, \; p \geq 0 \quad \ldots\ldots\ldots\ldots (c) \end{aligned}\right| \quad \ldots\ldots\ldots (57)$$

$$\bar{s}_1(x, 0, p) = \bar{s}_2(x, 0, p) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58)$$

$$\frac{\partial \bar{s}_1(x, 0, p)}{\partial y} = \delta \, \frac{\partial \bar{s}_2(x, 0, p)}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (59)$$

$$\lim_{r \to 0} r \frac{\partial \bar{s}_1}{\partial r} = \frac{-Qp}{2\pi T_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60)$$

where p is the variable obtained by using the Laplace transformation.

After separating the variables, it can be shown that

$$\frac{c}{p} \sin[f_n(x-a)] \exp[-y \sqrt{f_n^2 + p/\nu_1}] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (61)$$

and

$$\frac{c}{p} \sin[g_n(x-a)] \exp[\, y \sqrt{g_n^2 + p/\nu_2}] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (62)$$

are particular solutions of the differential equations in conditions (56) and (57), respectively, where c, f_n, and g_n are arbitrary constants.

By using conditions (56b) and (57b), it is easily seen that $f_n = g_n = n\pi/2a$.

Another particular solution is that of the steady-state case. As obtained by Hantush and Jacob [1955], this is given by

$$\frac{Q}{2\pi T_1 p} \sum_{n=0}^{\infty} \left[\frac{e^{-|y-y_1|/\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1}} \right] R_n$$

where $\epsilon_1 = (\frac{n\pi}{2a})^2 + \frac{p}{\nu_1}$ and

$$R_n = \sin\{ \frac{n\pi}{2a} (x-a) \} \sin\{ \frac{n\pi}{2a} (x_1 - a) \}$$

$$= \frac{1}{2} [\cos\{ \frac{n\pi}{2a} (x-x_1) \} - \cos\{ \frac{n\pi}{2a} (x+x_1-2a) \}]$$

Owing to the linearity and homogeneity of the differential equations, the above solutions can be combined linearly to obtain the following solutions:

$$\bar{s}_1 = \frac{c}{p} \sum_{n=1}^{\infty} [A_n e^{-y\sqrt{\epsilon_1}} + \frac{2}{2a/\pi} \frac{e^{-|y-y_1|\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1}}] R_n . \qquad (63)$$

$$\bar{s}_2 = \frac{c}{p} \sum_{n=1}^{\infty} [B_n e^{y\sqrt{\epsilon_2}}] R_n \qquad (64)$$

where $\epsilon_2 = (\frac{n\pi}{2a})^2 + \frac{p}{\nu_2}$, and $c = Q/2\pi T_1$.

Equations (63) and (64) satisfy all the transformed conditions except (58) and (59). The latter conditions are satisfied if the constants A_n and B_n in equations (63) and (64) are chosen as:

$$A_n = \frac{\pi}{a} \frac{(\sqrt{\epsilon_1} - \delta\sqrt{\epsilon_2})}{\sqrt{\epsilon_1}\,(\sqrt{\epsilon_1} + \delta\sqrt{\epsilon_2})} \; e^{-y_1\sqrt{\epsilon_1}}$$

$$B_n = \frac{2\pi}{a} \frac{e^{-y_1\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1} + \delta\sqrt{\epsilon_2}}$$

After substituting A_n and B_n in equations (63) and (64), the solutions become:

$$\bar{s}_1 = \frac{c}{p} \left(\frac{\pi}{a} \sum_{n=1}^{\infty} \left[\frac{2e^{-(y+y_1)\sqrt{\epsilon_1}}}{(\sqrt{\epsilon_1} + \delta\sqrt{\epsilon_2})} - \frac{e^{-(y+y_1)\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1}} \right.\right.$$

$$\left.\left. + \frac{e^{-|y-y_1|\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1}} \right] R_n \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (65)$$

$$\bar{s}_2 = \frac{c}{p} \left(\frac{2\pi}{a} \sum_{n=1}^{\infty} \left[\frac{e^{y\sqrt{\epsilon_2} - y_1\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1} + \delta\sqrt{\epsilon_2}} \right] R_n \right) \quad \ldots\ldots\ldots\ldots \quad (66)$$

The inverse transformations of the above equations can be obtained by using the method of integration in the complex plane (see appendix).

By collecting terms and simplifying, the transformed solution becomes:

$$\frac{s_1}{Q/2\pi T_1} = \sum_{n=0}^{\infty} \left(\frac{4\sqrt{\nu_1 t}}{a} e^{-(\frac{n\pi}{2a})^2 \nu_1 t} [V_1 - V_2] + F_n \right) R_n$$

$$+ \frac{1}{2} \left(\ln\left[\frac{\cosh[\frac{\pi}{2a}(y-y_1)] + \cos[\frac{\pi}{2a}(x+x_1)]}{\cosh[\frac{\pi}{2a}(y-y_1)] - \cos[\frac{\pi}{2a}(x-x_1)]} \right] \right.$$

$$\left. + \frac{1-\delta}{1+\delta} \ln\left[\frac{\cosh[\frac{\pi}{2a}(y+y_1)] + \cos[\frac{\pi}{2a}(x+x_1)]}{\cosh[\frac{\pi}{2a}(y+y_1)] - \cos[\frac{\pi}{2a}(x-x_1)]} \right] \right) \quad \ldots \ldots \ldots \ldots (67)$$

$$\frac{s_2}{Q/2\pi T_1} = \sum_{n=0}^{\infty} \left(\frac{4\sqrt{\nu_1 t}}{a} e^{-\left(\frac{n\pi}{2a}\right)^2 \nu_1 t} [W_1 - W_2] \right) R_n$$

$$+ \frac{1}{2} \left[1 + \frac{1-\delta}{1+\delta} \right] \ln \left[\frac{\cosh\left[\frac{\pi}{2a}(y-y_1)\right] + \cos\left[\frac{\pi}{2a}(x+x_1)\right]}{\cosh\left[\frac{\pi}{2a}(y-y_1)\right] - \cos\left[\frac{\pi}{2a}(x-x_1)\right]} \right] .$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (68)$$

where

$$V_1 = \int_0^{\frac{n\pi}{2a}\sqrt{|\nu_1 - \nu_2| t}} \frac{\exp\left(\beta^2 - (y+y_1)\beta/\sqrt{\nu_1 t}\right)}{\left[\beta^2 - \left(\frac{n\pi}{2a}\right)^2 \nu_1 t\right]} \cdot$$

$$\frac{\left[\delta\sqrt{\nu_1/\nu_2}\sqrt{\left(\frac{n\pi}{2a}\right)^2 |\nu_1 - \nu_2| t - \beta^2}\right] \beta \, d\beta}{\left[\beta^2 + \delta^2 (\nu_1/\nu_2) \left\{\left(\frac{n\pi}{2a}\right)^2 |\nu_1 - \nu_2| t - \beta^2\right\}\right]} .$$

$$V_2 = \int_0^{\infty} \frac{e^{-\beta^2}}{\left[\beta^2 + \left(\frac{n\pi}{2a}\right)^2 \nu_1 t\right]} \cdot$$

$$\frac{\cos\left\{(y+y_1)\beta/\sqrt{\nu_1 t}\right\} \beta \, d\beta}{\left[\beta + \delta\sqrt{\nu_1/\nu_2}\sqrt{\left(\frac{n\pi}{2a}\right)^2 |\nu_1 - \nu_2| t + \beta^2}\right]} .$$

$$W_1 = \int_0^{\frac{n\pi}{2a}\sqrt{|\nu_1 - \nu_2|t}} \exp(\beta^2 - y_1\beta / \sqrt{\nu_1 t}\ )\ .$$

$$\left[\frac{\beta \sin\{ -\dfrac{y}{\sqrt{\nu_2 t}} \sqrt{(\frac{n\pi}{2a})^2 \ |\nu_1 - \nu_2|t - \beta^2}\ \}}{(\beta^2 - (\frac{n\pi}{2a})^2\ \nu_1 t)\ (\beta^2 + \delta^2\{\nu_1/\nu_2\}\{(\frac{n\pi}{2a})^2\ |\nu_1-\nu_2|t-\beta^2\}\ )} \right.$$

$$+ \frac{\delta \sqrt{\dfrac{\nu_1}{\nu_2}} \sqrt{(\frac{n\pi}{2a})^2\ |\nu_1-\nu_2|t-\beta^2}}{(\beta^2 - (\frac{n\pi}{2a})^2\ \nu_1 t)\ (\beta^2 + \delta^2\{\nu_1/\nu_2\}\{(\frac{n\pi}{2a})^2\ |\nu_1-\nu_2|t-\beta^2\}\ )}\ .$$

$$\left. \cos\{ -\frac{y}{\sqrt{\nu_2 t}} \sqrt{(\frac{n\pi}{2a})^2\ |\nu_1-\nu_2|t-\beta^2}\ \} \right]\ \beta\ d\beta.$$

and

$$W_2 = \int_0^{\infty} \frac{e^{-\beta^2} \cos(\ \dfrac{y_1\beta}{\sqrt{\nu_1 t}} - \dfrac{y}{\sqrt{\nu_2 t}} \sqrt{(\frac{n\pi}{2a})^2\ |\nu_1-\nu_2|t-\beta^2}\ \}\ \beta\ d\beta}{(\beta^2 + (\frac{n\pi}{2a})^2\ \nu_1 t)\ (\beta + \delta\sqrt{\nu_1/\nu_2}\ \sqrt{(\frac{n\pi}{2a})^2 |\nu_1-\nu_2|t+\beta^2}\ )}$$

$$F_n = \frac{1}{n} [\exp\{-(y+y_1)(\frac{n\pi}{2a})\}\operatorname{erfc}\{(\frac{n\pi}{2a})\sqrt{\nu_1 t} - \frac{(y+y_1)}{\sqrt{4\nu_1 t}}\}$$

$$+ \exp\{(y+y_1)(\frac{n\pi}{2a})\}\operatorname{erfc}\{(\frac{n\pi}{2a})\sqrt{\nu_1 t} + \frac{(y+y_1)}{\sqrt{4\nu_1 t}}\}$$

$$- \exp\{-|y-y_1|(\frac{n\pi}{2a})\}\operatorname{erfc}\{(\frac{n\pi}{2a})\sqrt{\nu_1 t} - \frac{|y-y_1|}{\sqrt{4\nu_1 t}}\}$$

$$- \exp\{|y-y_1|(\frac{n\pi}{2a})\}\operatorname{erfc}\{(\frac{n\pi}{2a})\sqrt{\nu_1 t} + \frac{|y-y_1|}{\sqrt{4\nu_1 t}}\}]$$

Equations (67) and (68) represent the general solution of the problem. The integrals V_1, V_2, W_1, and W_2 are rapidly convergent; however, they do not lend themselves readily to easy tabulation.

Steady-state solution -- As pumping continues indefinitely (that is, as $t \to \infty$), the equations (67) and (68) reduce to the steady-state solution:

$$\frac{s_1}{Q/2\pi T_1} = \frac{1}{2} \left(\ln\left[\frac{\cosh[\frac{\pi}{2a}(y-y_1)] + \cos[\frac{\pi}{2a}(x+x_1)]}{\cosh[\frac{\pi}{2a}(y-y_1)] - \cos[\frac{\pi}{2a}(x-x_1)]} \right] \right.$$

$$\left. + \frac{1-\delta}{1+\delta} \ln\left[\frac{\cosh[\frac{\pi}{2a}(y+y_1)] + \cos[\frac{\pi}{2a}(x+x_1)]}{\cosh[\frac{\pi}{2a}(y+y_1)] - \cos[\frac{\pi}{2a}(x-x_1)]} \right] \right) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots (69)$$

$$\frac{s_2}{Q/2\pi T_1} = \frac{1}{2} \left[1 + \frac{1-\delta}{1+\delta} \right] \ln\left[\frac{\cosh[\frac{\pi}{2a}(y-y_1)] + \cos[\frac{\pi}{2a}(x+x_1)]}{\cosh[\frac{\pi}{2a}(y-y_1)] - \cos[\frac{\pi}{2a}(x-x_1)]} \right] \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots (70)$$

<u>Solution of a special case</u> -- It is of interest to note that if $\delta \to 1$ (that is, if the two regions form a single region of uniform conductivity), either equation (69) or (70) becomes

$$\frac{s}{Q/2\pi T_1} = \frac{1}{2} \ln\left[\frac{\cosh[\frac{\pi}{2a}(y-y_1)] + \cos[\frac{\pi}{2a}(x+x_1)]}{\cosh[\frac{\pi}{2a}(y-y_1)] - \cos[\frac{\pi}{2a}(x-x_1)]} \right] \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots (71)$$

which is the steady-state solution for a uniform infinite strip of nonleaky aquifer [Hantush and Jacob, 1954].

Case B. Zero Drawdown on One Boundary and Zero Flux Across the Other

When the flow system set up in problem 2, Case A is reflected positively in the y-axis (that is, if another well is supposed to exist at $(-x_1, y_1)$), there will be no flow across the y-axis.

The drawdown produced by an actual well with center at (x_1, y_1) is given by equations (67) and (68); that produced by a hypothetical well with center at $(-x_1, y_1)$ is given by equations (67) and (68) with the sign of x_1 reversed.

The drawdown due to both of these wells (actual and hypothetical) will give the drawdown produced by a single well placed at (x_1, y_1) in an infinite strip of width a, whose left boundary (fig. 2, case B) is impermeable therefore, addition of the right-hand members of (67) and (68) to themselves, with x_1 replaced by $-x_1$, will, after simplification, give the required drawdown distribution, which is:

$$\frac{s_1}{Q/2\pi T_1} = \sum_{n=0}^{\infty} \left(- \frac{8\sqrt{\nu_1 t}}{a} \exp\left[- \left(\frac{n\pi}{2a}\right)^2 \nu_1 t \right] \{V_1 - V_2\} + 2F_n \right) \cdot$$

$$\sin\{[(\frac{\pi}{2a})(2n-1)](x-a)\} \cos\{[(\frac{\pi}{2a})(2n-1)]x_1\}$$

$$+ \frac{1}{2} \left(\ln\left[\frac{\cosh[\frac{\pi}{2a}(y-y_1)] + \cos[\frac{\pi}{2a}(x+x_1)]}{\cosh[\frac{\pi}{2a}(y-y_1)] - \cos[\frac{\pi}{2a}(x-x_1)]} \cdot \right. \right.$$

$$\left. \frac{\cosh[\frac{\pi}{2a}(y-y_1)] + \cos[\frac{\pi}{2a}(x-x_1)]}{\cosh[\frac{\pi}{2a}(y-y_1)] - \cos[\frac{\pi}{2a}(x+x_1)]} \right]$$

$$+ \frac{1-\delta}{1+\delta} \ln\left[\frac{\cosh[\frac{\pi}{2a}(y+y_1)] + \cos[\frac{\pi}{2a}(x+x_1)]}{\cosh[\frac{\pi}{2a}(y+y_1)] - \cos[\frac{\pi}{2a}(x-x_1)]} \cdot \right.$$

$$\left. \left. \frac{\cosh[\frac{\pi}{2a}(y+y_1)] + \cos[\frac{\pi}{2a}(x-x_1)]}{\cosh[\frac{\pi}{2a}(y+y_1)] - \cos[\frac{\pi}{2a}(x+x_1)]} \right] \right) \quad \ldots\ldots\ldots \quad (72)$$

$$\frac{s_2}{Q/2\pi T_1} = \sum_{n=0}^{\infty} \Big(- \frac{8\sqrt{\nu_1 t}}{a} \exp\Big[- \Big(\frac{n\pi}{2a}\Big)^2 \nu_1 t \Big] (W_1 - W_2) \cdot$$

$$\sin\Big\{\Big[\Big(\frac{\pi}{2a}\Big)(2n-1)\Big](x-a)\Big\} \cos\Big\{\Big[\Big(\frac{\pi}{2a}\Big)(2n-1)\Big]x_1\Big\}$$

$$+ \frac{1}{2}\Big[1 - \frac{1-\delta}{1+\delta}\Big] \ln\Big[\frac{\cosh[\frac{\pi}{2a}(y-y_1)] + \cos[\frac{\pi}{2a}(x+x_1)]}{\cosh[\frac{\pi}{2a}(y-y_1)] - \cos[\frac{\pi}{2a}(x-x_1)]} \cdot$$

$$\frac{\cosh[\frac{\pi}{2a}(y-y_1)] + \cos[\frac{\pi}{2a}(x-x_1)]}{\cosh[\frac{\pi}{2a}(y-y_1)] - \cos[\frac{\pi}{2a}(x+x_1)]} \Big] \quad \ldots\ldots\ldots\ldots (73)$$

Steady-state solution -- The steady-state solution can be developed by a procedure similar to that used in solving case A:

$$\frac{s_1}{Q/2\pi T_1} = \frac{1}{2}\left(\ln\left[\frac{\cosh\left[\frac{\pi}{2a}(y-y_1)\right] + \cos\left[\frac{\pi}{2a}(x+x_1)\right]}{\cosh\left[\frac{\pi}{2a}(y-y_1)\right] - \cos\left[\frac{\pi}{2a}(x-x_1)\right]} \cdot \frac{\cosh\left[\frac{\pi}{2a}(y-y_1)\right] + \cos\left[\frac{\pi}{2a}(x-x_1)\right]}{\cosh\left[\frac{\pi}{2a}(y-y_1)\right] - \cos\left[\frac{\pi}{2a}(x+x_1)\right]} \right] + \frac{1-\delta}{1+\delta} \ln\left[\frac{\cosh\left[\frac{\pi}{2a}(y+y_1)\right] + \cos\left[\frac{\pi}{2a}(x+x_1)\right]}{\cosh\left[\frac{\pi}{2a}(y+y_1)\right] - \cos\left[\frac{\pi}{2a}(x-x_1)\right]} \cdot \frac{\cosh\left[\frac{\pi}{2a}(y+y_1)\right] + \cos\left[\frac{\pi}{2a}(x-x_1)\right]}{\cosh\left[\frac{\pi}{2a}(y+y_1)\right] - \cos\left[\frac{\pi}{2a}(x+x_1)\right]} \right] \right) \quad \cdots\cdots\cdots (74)$$

$$\frac{s_2}{Q/2\pi T_1} = \frac{1}{2}\left[1 - \frac{1-\delta}{1+\delta} \right] \ln\left[\frac{\cosh\left[\frac{\pi}{2a}(y-y_1)\right] + \cos\left[\frac{\pi}{2a}(x+x_1)\right]}{\cosh\left[\frac{\pi}{2a}(y-y_1)\right] - \cos\left[\frac{\pi}{2a}(x-x_1)\right]} \cdot \frac{\cosh\left[\frac{\pi}{2a}(y-y_1)\right] + \cos\left[\frac{\pi}{2a}(x-x_1)\right]}{\cosh\left[\frac{\pi}{2a}(y-y_1)\right] - \cos\left[\frac{\pi}{2a}(x+x_1)\right]} \right] \quad \cdots\cdots\cdots\cdots (75)$$

Case C. Zero Flux Across Both Boundaries

This case is similar to case A except that the boundaries of the infinite strip are maintained at vanishing flux. The solution of the following system of equations is required:

$$\left.\begin{aligned}
&\frac{\partial^2 s_1}{\partial x^2} + \frac{\partial^2 s_1}{\partial y^2} = \frac{1}{\nu_1}\frac{\partial s_1}{\partial t} \quad \ldots\ldots \text{(a)} \\
&\frac{\partial s_1(\pm a, y, t)}{\partial x} = 0, \; t \geqq 0 \quad \ldots \text{(b)} \\
&s_1(x, \infty, t) = 0, \; t \geqq 0 \quad \ldots\ldots \text{(c)}
\end{aligned}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots (76)$$

$$\left.\begin{aligned}
&\frac{\partial^2 s_2}{\partial x^2} + \frac{\partial^2 s_2}{\partial y^2} = \frac{1}{\nu_2}\frac{\partial s_2}{\partial t} \quad \ldots\ldots \text{(a)} \\
&\frac{\partial s_2(\pm a, y, t)}{\partial x} = 0, \; t \geqq 0 \quad \ldots \text{(b)} \\
&s_2(x, -\infty, t) = 0, \; t \geqq 0 \quad \ldots\ldots \text{(c)}
\end{aligned}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots (77)$$

$$s_1(x, y, 0) = s_2(x, y, 0) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (78)$$

$$\frac{\partial s_1(x, 0, t)}{\partial y} = \delta \frac{\partial s_2(x, 0, t)}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (79)$$

$$s_1(x,0,t) = s_2(x,0,t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (80)$$

$$\underset{r \to 0}{\text{Limit}}\ r \frac{\partial s_1}{\partial r} = \frac{-Q}{2\pi T_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (81)$$

By using condition (78), the Laplace transform of the boundary value problem is:

$$\left.\begin{array}{ll} \dfrac{\partial^2 \bar{s}_1}{\partial x^2} + \dfrac{\partial^2 \bar{s}_1}{\partial y^2} - \dfrac{p\bar{s}_1}{\nu_1} = 0 \quad \ldots\ldots\ldots\ldots & \text{(a)} \\ \dfrac{\partial \bar{s}_1(\pm a,y,p)}{\partial x} = 0, \ p \geq 0 \quad \ldots\ldots\ldots\ldots & \text{(b)} \\ \bar{s}_1(x,\infty,p) = 0, \ p \geq 0 \quad \ldots\ldots\ldots\ldots & \text{(c)} \end{array}\right| \quad \ldots\ldots\ldots \quad (82)$$

$$\left.\begin{array}{ll} \dfrac{\partial^2 \bar{s}_2}{\partial x^2} + \dfrac{\partial^2 \bar{s}_2}{\partial y^2} - \dfrac{p\bar{s}_2}{\nu_2} = 0 \quad \ldots\ldots\ldots\ldots & \text{(a)} \\ \dfrac{\partial \bar{s}_2(\pm a,y,p)}{\partial x} = 0, \ p \geq 0 \quad \ldots\ldots\ldots\ldots & \text{(b)} \\ \bar{s}_2(x,-\infty,p) = 0, \ p \geq 0 \quad \ldots\ldots\ldots\ldots & \text{(c)} \end{array}\right| \quad \ldots\ldots\ldots \quad (83)$$

$$\bar{s}_1(x,0,p) = \bar{s}_2(x,0,p) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (84)$$

$$\frac{\partial \bar{s}_1(x,0,p)}{\partial y} = \delta \frac{\partial \bar{s}_2(x,0,p)}{\partial y} \qquad \ldots\ldots\ldots\ldots (85)$$

$$\bar{s}_1(x,y,0) = \bar{s}_2(x,y,0) = 0 \qquad \ldots\ldots\ldots\ldots (86)$$

$$\underset{r\to 0}{\text{Limit}}\ r \frac{\partial \bar{s}_1}{\partial r} = \frac{-Qp}{2\pi T_1} \qquad \ldots\ldots\ldots\ldots (87)$$

where the terms are as defined before. It can be shown that $e^{-y\sqrt{\epsilon_1}}\,G_n$ and $e^{y\sqrt{\epsilon_2}}\,G_n$, where $G_n = \cos\{\frac{n\pi}{2a}(x_1-a)\}$. $\cos\{\frac{n\pi}{2a}(x-a)\}$ are particular solutions of equations (82a) and (83a) that satisfy conditions (82b), (82c), (83b), and (83c). A known solution of (82a) and (83a) satisfying the above conditions, as well as condition (87) [Hantush and Jacob, 1954], is

$$\frac{c}{p}\frac{\pi}{2a}\left\{\frac{e^{-|y-y_1|\sqrt{p/\nu_1}}}{\sqrt{p/\nu_1}} + 2\sum_{n=1}^{\infty}\frac{e^{-|y-y_1|\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1}}\right\} G_n$$

where c has been determined to be equal to $Q/2\pi T_1$. As in problem 2, case A, the solutions can be combined linearly to obtain

$$\bar{s}_1 = \frac{c}{p}\{\sum_{n=1}^{\infty}[A_n e^{-y\sqrt{\epsilon_1}} + \frac{\pi}{a}\frac{e^{-|y-y_1|\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1}}]\,G_n$$

$$+ A_0\, e^{-y\sqrt{p/\nu_1}} + \frac{\pi}{2a}\frac{e^{-|y-y_1|\sqrt{p/\nu_1}}}{\sqrt{p/\nu_1}}\} \qquad \ldots\ldots\ldots\ldots (88)$$

$$\bar{s}_2 = \frac{c}{p}\{\sum_{n=1}^{\infty}[B_n\, e^{y\sqrt{\epsilon_2}}]\,G_n + B_0\, e^{y\sqrt{p/\nu_2}}\} \qquad \ldots\ldots\ldots\ldots (89)$$

A_0, A_n, B_0 and B_n are determined by using the remaining boundary conditions:

$$A_0 = \frac{\pi}{2a}\frac{e^{-y_1\sqrt{p/\nu_1}}}{\sqrt{p/\nu_1}}\{\frac{\sqrt{p/\nu_1} - \delta\sqrt{p/\nu_2}}{\sqrt{p/\nu_1} + \delta\sqrt{p/\nu_2}}\}$$

$$A_n = \frac{\pi}{a}\frac{e^{-y_1\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1}}\{\frac{\sqrt{\epsilon_1} - \delta\sqrt{\epsilon_2}}{\sqrt{\epsilon_1} + \delta\sqrt{\epsilon_2}}\}$$

$$B_0 = \frac{\pi}{2a} \left\{ \frac{2e^{-y_1\sqrt{p/\nu_1}}}{\sqrt{p/\nu_1} + \delta\sqrt{p/\nu_2}} \right\}$$

$$B_n = \frac{\pi}{a} \left\{ \frac{2e^{-y_1\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1} + \delta\sqrt{\epsilon_2}} \right\}$$

Substituting the above in equations (88) and (89), one obtains:

$$\bar{s}_1 = \frac{c}{p} \left\{ \frac{\pi}{a} \sum_{n=1}^{\infty} \left[\frac{2e^{-(y+y_1)\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1} + \delta\sqrt{\epsilon_2}} - \frac{e^{-(y+y_1)\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1}} \right.\right.$$

$$\left. + \frac{e^{-|y-y_1|\sqrt{\epsilon_1}}}{\sqrt{\epsilon_1}} \right] G_n$$

$$+ \frac{\pi}{a} \left[\frac{2e^{-(y+y_1)\sqrt{p/\nu_1}}}{\sqrt{p/\nu_1} + \delta\sqrt{p/\nu_2}} - \frac{e^{-(y+y_1)\sqrt{p/\nu_1}}}{\sqrt{p/\nu_1}} \right.$$

$$\left.\left. + \frac{e^{-|y-y_1|\sqrt{p/\nu_1}}}{\sqrt{p/\nu_1}} \right] \right\} \qquad \text{.............................} \qquad (90)$$

$$\bar{s}_2 = \frac{c}{p} \left\{ \frac{\pi}{a} \sum_{n=1}^{\infty} \left[\frac{2e^{(y\sqrt{\epsilon_2} + y_1\sqrt{\epsilon_2})}}{\sqrt{\epsilon_1} + \delta\sqrt{\epsilon_2}} \right] G_n \right.$$

$$\left. + \frac{\pi}{a} \frac{e^{(y\sqrt{p/\nu_2} - y_1\sqrt{p/\nu_1})}}{\sqrt{p/\nu_1} + \delta\sqrt{p/\nu_2}} \right\} \qquad (91)$$

As the inverse transforms of the series have been determined (problem 1, case A), and the inverse transforms of the remaining terms are tabulated, one can write the solution in the form

$$\frac{s_1}{Q/2\pi T_1} = \sum_{n=1}^{\infty} \left[\frac{4\sqrt{\nu_1 t}}{a} e^{-\left(\frac{n\pi}{2a}\right)^2 \nu_1 t} \{V_1 - V_2\} + F_n \right.$$

$$\left. + \frac{2}{n} \left\{ e^{-\left(\frac{n\pi}{2a}\right)|y-y_1|} + \frac{1-\delta}{1+\delta} e^{-\left(\frac{n\pi}{2a}\right)(y+y_1)} \right\} \right] G_n$$

$$+ H \qquad (92)$$

$$\frac{s_2}{Q/2\pi T_1} = \sum_{n=1}^{\infty} \Big[\frac{4\sqrt{\nu_1 t}}{a} \, e^{-\left(\frac{n\pi}{2a}\right)^2 \nu_1 t} \{W_1 - W_2\}$$

$$+ \frac{4}{n} \, \frac{e^{\left(\frac{n\pi}{2a}\right)(y-y_1)}}{1+\delta} \Big] G_n + J \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (93)$$

where

$$H = \frac{\pi}{2a} \Big[\sqrt{4\nu_1 t/\pi} \; e^{-\frac{|y-y_1|^2}{4\nu_1 t}} + \frac{\sqrt{\nu_2} - \delta\sqrt{\nu_1}}{\sqrt{\nu_2} + \delta\sqrt{\nu_1}} \cdot$$

$$\left\{ \sqrt{4\nu_1 t/\pi} \; e^{-\frac{(y+y_1)^2}{4\nu_1 t}} - (y+y_1)\,\mathrm{erfc}\left(\frac{y+y_1}{\sqrt{4\nu_1 t}}\right) \right\}$$

$$- |y-y_1|\,\mathrm{erfc}\left(\frac{|y-y_1|}{\sqrt{4\nu_1 t}}\right) \Big]$$

and

$$J = \frac{\pi}{a} \left[\frac{2\sqrt{\nu_1 \nu_2 t/\pi}}{\sqrt{\nu_2} + \delta\sqrt{\nu_1}} \exp\left(- \frac{\{y_1\sqrt{\nu_2} - y\sqrt{\nu_1}\}^2}{4\nu_1\nu_2 t} \right) \right.$$

$$\left. - \frac{y_1\sqrt{\nu_2} - y\sqrt{\nu_1}}{\sqrt{\nu_2} + \delta\sqrt{\nu_1}} \operatorname{erfc}\left(\frac{y_1\sqrt{\nu_2} - y\sqrt{\nu_1}}{\sqrt{4\nu_1\nu_2 t}} \right) \right]$$

As $t \to \infty$, s_1 and s_2 approach ∞; this is as one would expect, there being no source of recharge.

ADAPTATION OF SOLUTIONS TO WATER-TABLE FLOW SYSTEMS

The solutions obtained in the above analyses are for artesian flow systems. They can, however, be adapted for water-table flow systems if the maximum drawdown is small compared to the original depth of flow. According to Jacob [1950], if $s < 0.25h_0$, h_0 being the original depth of flow (original thickness of water-bearing material), and provided the storage coefficient remains uniform, the artesian flow formulas may be used with sufficient accuracy if s is replaced by $s - s^2/2h_0$. Thus, if in all the formulas developed in the preceding analyses the values of s and s_w are respectively replaced by $(s - s^2/2h_0)$ and $(s_w - s_w^2/2h_0)$, the resulting expression will describe drawdown distribution for water-table flow systems; the transmissivity, T, in these formulas will be that of the original depth of flow.

P A R T I I I

SOLUTIONS FOR SPECIAL CASES

Applications of the solutions for problem 1, case A, flow toward a slit at constant drawdown, require that the so-called formation constants and the dimensions of the aquifer be determined. Values which have been determined for the Roswell ground-water reservoir, Roswell, New Mexico (Hantush, 1955), are:

L (distance from the slit to the interface between contiguous regions) = 15 miles

S_1 (Storage coefficient in region 1) = 10^{-5}

S_2 (Storage coefficient in region 2) = 0.05

T_1 (Transmissibility in region 1) = T_2 (transmissibility in region 2) = 0.12 ft.^{2}/sec.

Figures 3 and 4 are graphic representations of values found in table 1. Table 1 was prepared using the above constants in equations (15) and (21). Values in table 1 represent the variation of drawdown at the interface between contiguous regions with respect to time.

Figure 5 is the graphic representation of values found in table 2. Table 2 was prepared by substituting the above constants in equation (20). The values in table 2 represent the variation of drawdown with distance from the slit after 605 days of flow.

The steady-state solutions (problem 2, cases A and B) equations (69), (70), (74), and (75), have been tabulated by the I.B.M. 704 data processing system for several values of the parameters: x_1/a, y_1/a, and δ. The values, $s_1/Q/2\pi T_1$ and $s_2/Q/2\pi T_1$ as a function of x/a and y/a, being too numerous to include, are kept on file at the New Mexico Institute of Mining and Technology, along with a copy of the program used to obtain them. The program tabulates drawdown values for any desired number of grid points $(x/a, y/a)$, up to 2500, for any given x_1/a, y_1/a and δ. It also produces decimal cards which can be used to plot equipotential contours.

Figures 6, 7, and 8 are graphical representations of the drawdown distribution (infinite strip bounded by two streams) for the flow to a well with center at $(x_1/a, y_1/a)$ equal to (.5, .5) with δ equal to 1/9, 1/2 and 1, respectively. Figure 9 represents cross sections of figures 6, 7, and 8, parallel to the y-axis and figure 10 represents cross sections parallel to the x-axis.

Figure 11 was made by superimposing figures 6, 7, and 8 in order to illustrate the variation of drawdown with δ.

Figure 12 is similar to figures 6, 7, and 8, except that the flow is to a well with center at $(x_1/a, y_1/a)$ equal to (.5, .2) and δ equal to 10. Cross sections of this case are presented in figures 13 and 14.

Figures 15, 18, and 21 are graphical representations of the drawdown distribution (infinite strip having one boundary maintained at zero drawdown and zero flux across the other) for the flow to a well with center at $(x_1/a, y_1/a)$ equal to (.5, .5) and $\delta = 1/9$, $1/2$, and 1, respectively. Cross-sections of these figures are represented by figures 16, 17, 19, 20, 22, and 23.

Figure 18 illustrates the hypothetical "image well" which is used to determine the analytical solution of the problem.

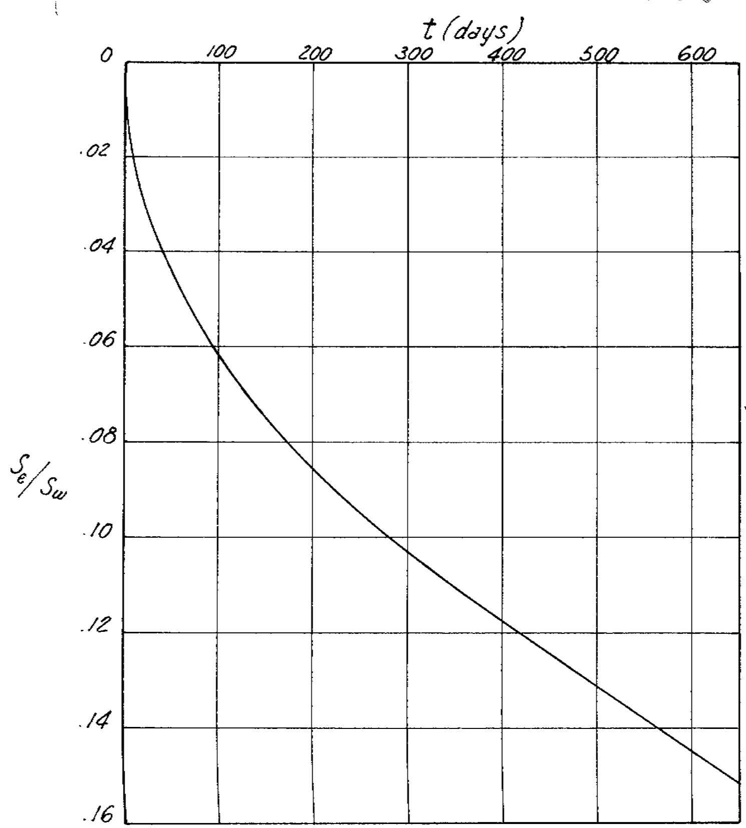
Table 1. -- Drawdown values (s_L/s_w) for $x = L = 15$ mi., $\delta = 1$, and $\gamma = -.97211$.

Equation	s_L/s_w values for t(days) equal to					
	50.00	105.03	186.73	308.67	420.13	604.99
15	.04283	.06291	.08255	.1043	.1200	...
21	...	...	.08286	.1045	.1203	.1454
	s_L/s_w values for t(years) equal to					
	25.00	50.00	100.00	200.00	300.00	500.00
21	0.365	0.502	0.599	0.687	0.739	0.787

Table 2. -- Drawdown values (s/s_w) for $t = 605$ days, $L = 15$ mi., and $\gamma = -.97211$ (see equation 21.)

s/s_w values for x/L equal to						
1.000	1.050	1.100	1.150	1.200	1.250	1.350
.1415	.1026	.07124	.04776	.03052	.01866	.00602

Fig. 3 -- Time-drawdown variation (zero to 605 days) at the interface of two contiguous regions drained by a slit maintained at a constant drawdown.



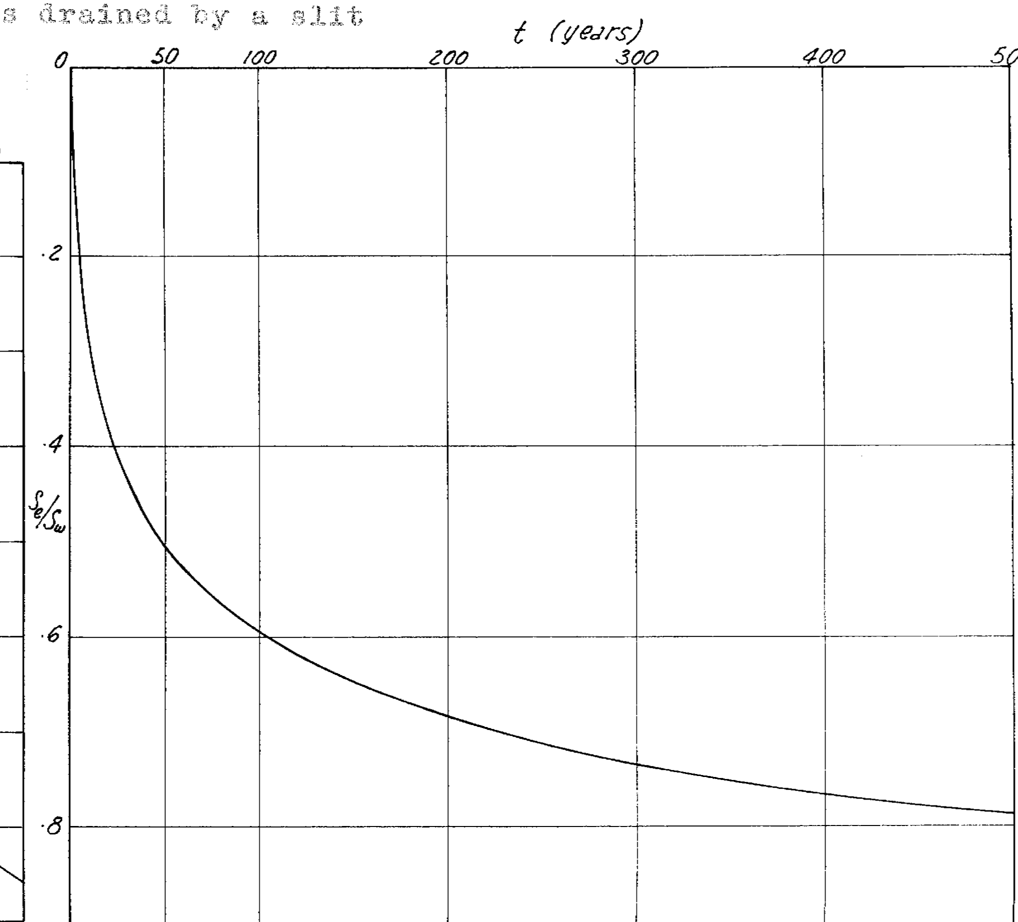


Fig. 4 -- Time-drawdown variation (zero to 500 years) at the interface of two contiguous regions drained by a slit maintained at a constant drawdown.

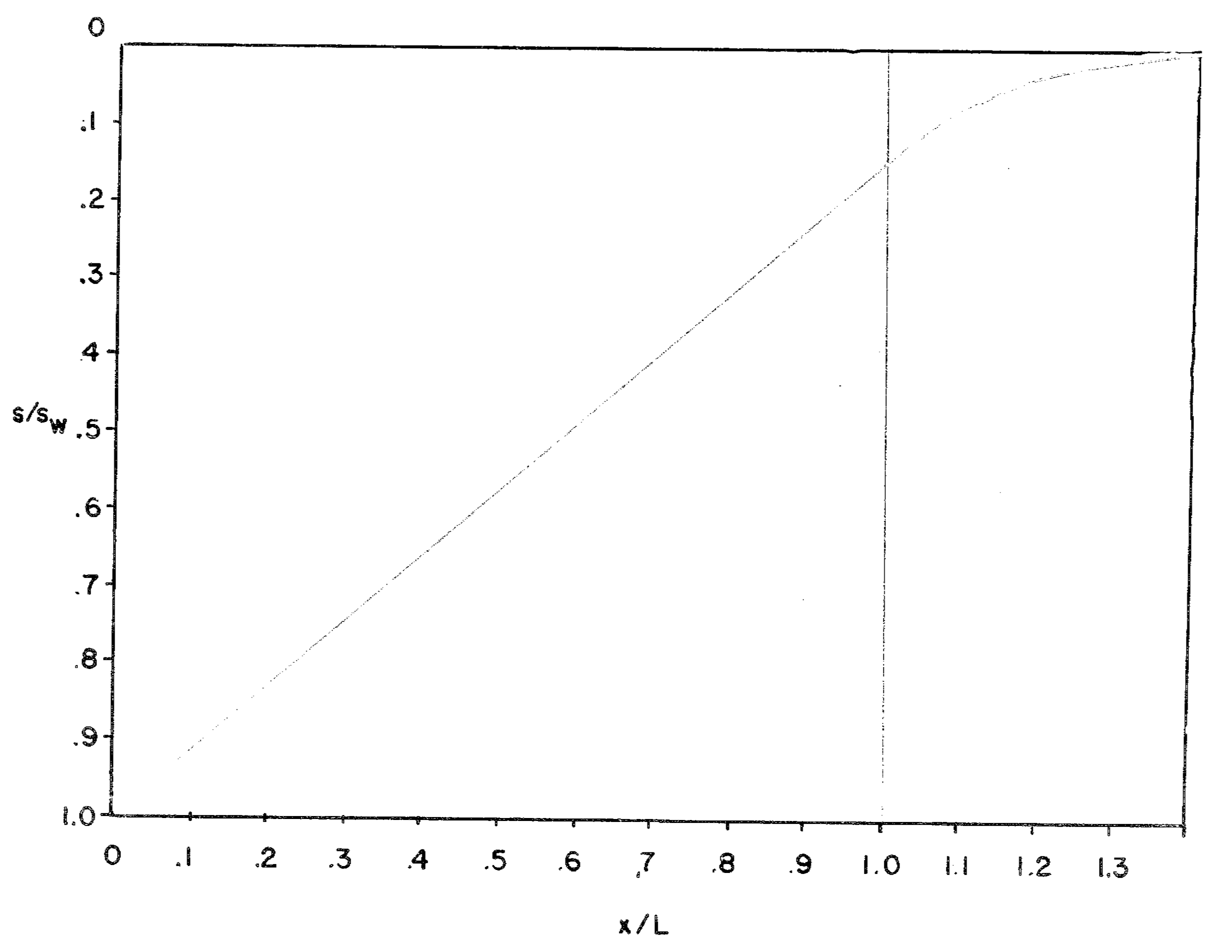


Fig. 5 -- Distance-drawdown variation after .05 days of continuous discharge of a slit (Roswell Basin).

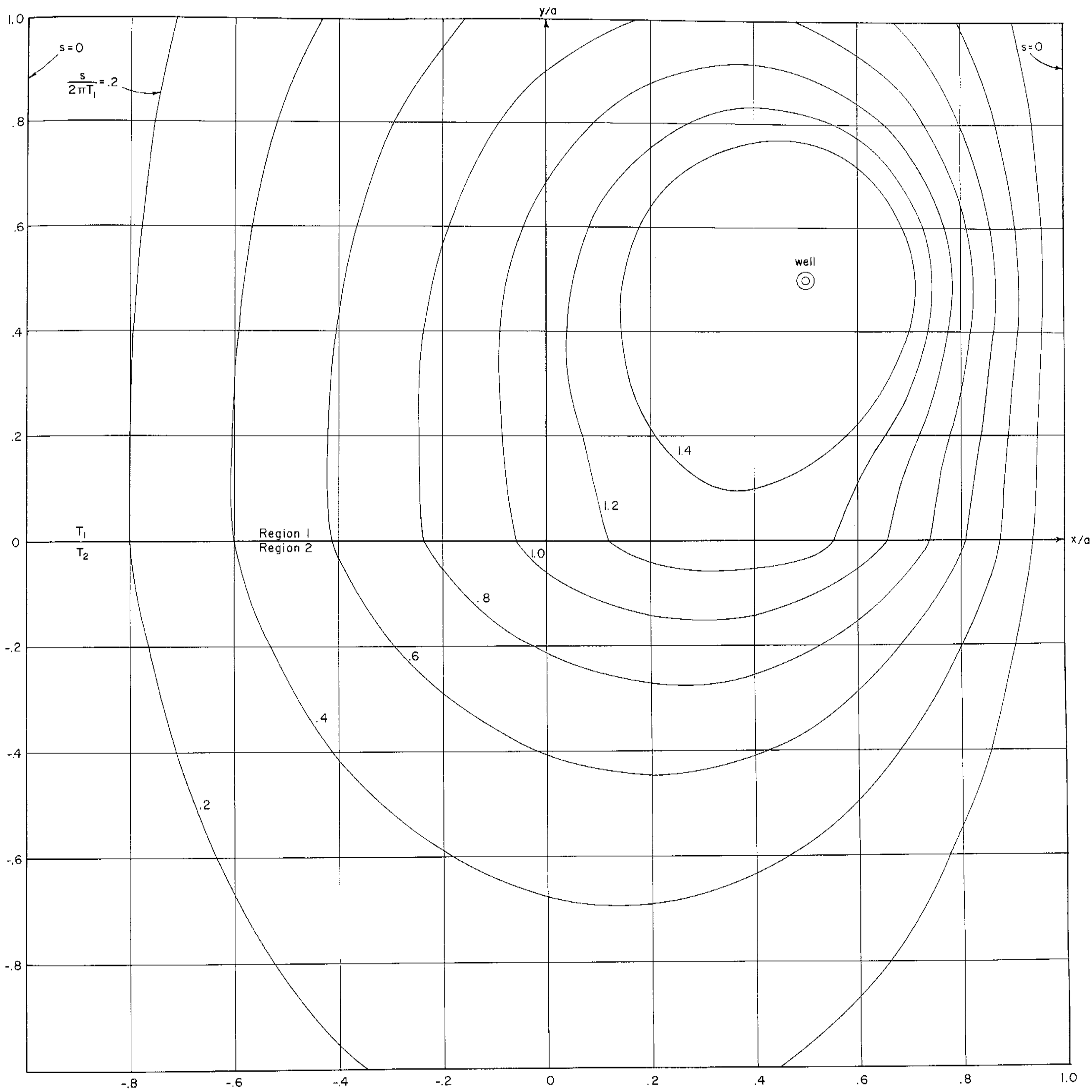


Fig. 6 -- Steady-state equi-drawdown lines of an infinite strip of aquifer having parallel boundaries maintained at zero drawdown, with well coordinates $x_1/a = y_1/a = 0.5$ and $\delta = 1/9$.

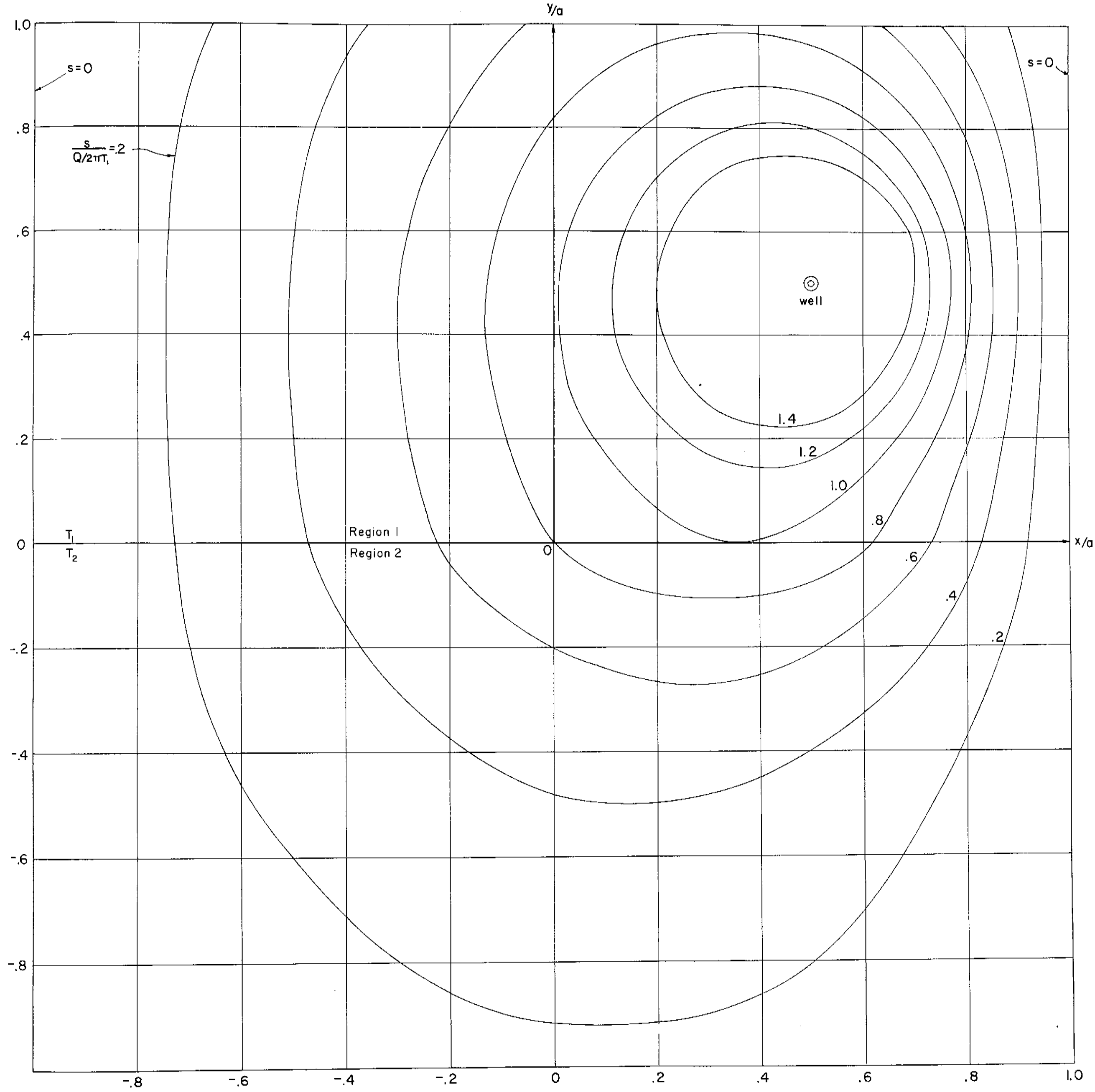


Fig. 7 -- Steady-state equi-drawdown lines of an infinite strip of aquifer having parallel boundaries maintained at zero drawdown, with well coordinates $x_1/a = y_1/a = 0.5$ and $\delta = 1/2$.

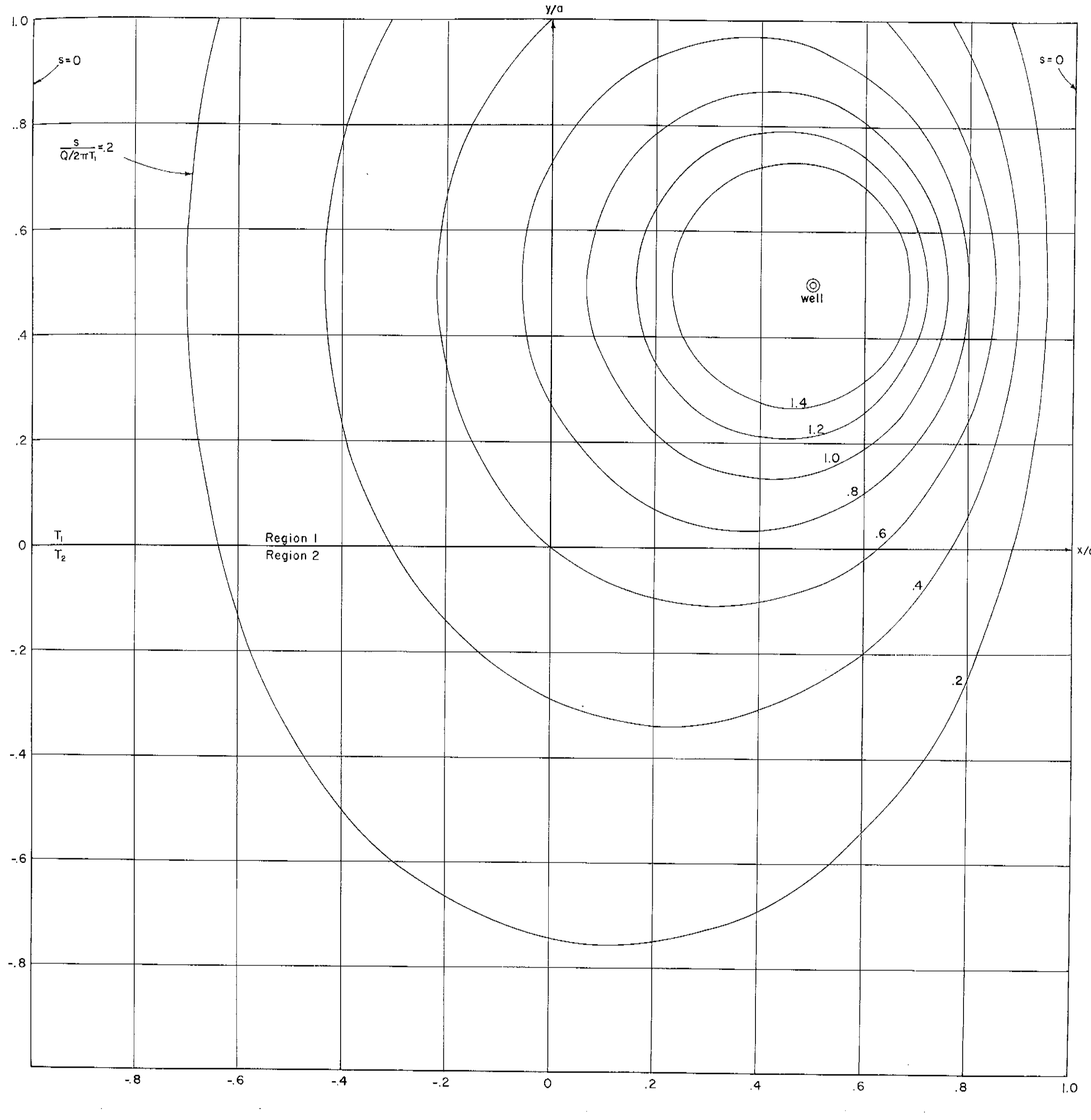


Fig. 8 — Steady-state equi-drawdown lines of an infinite strip of aquifer having parallel boundaries maintained at zero drawdown, with well coordinates $x_1/a = y_1/a = 0.5$ and $\delta = 1$.

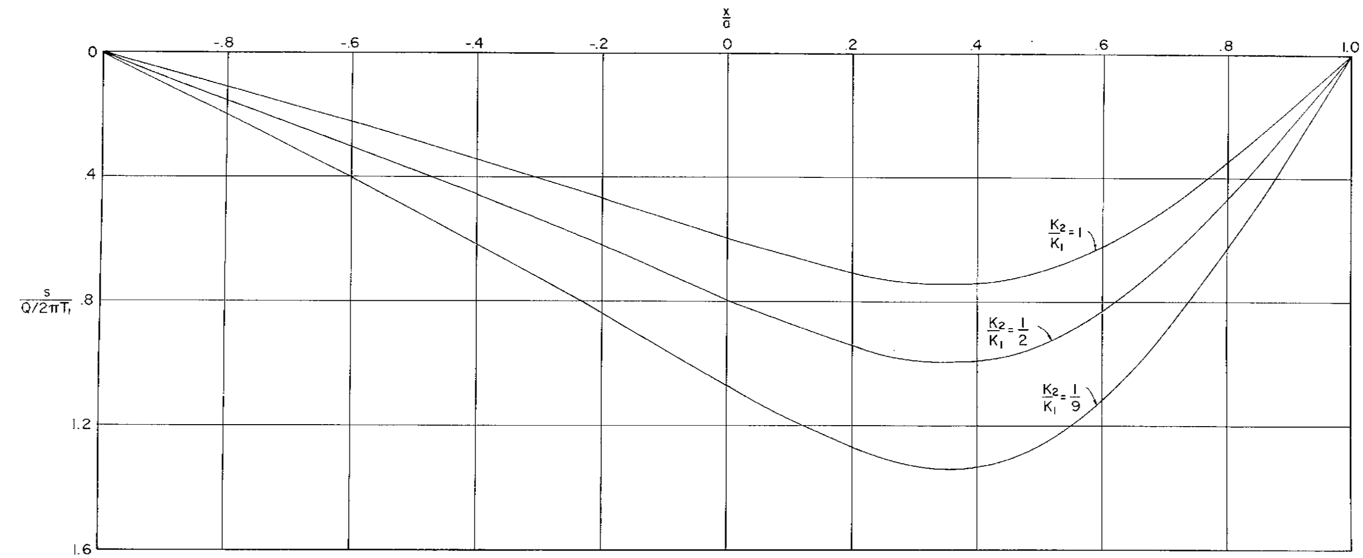


Fig. 9 -- Drawdown variation along the interface for different values of δ (Problem 2, Case A).

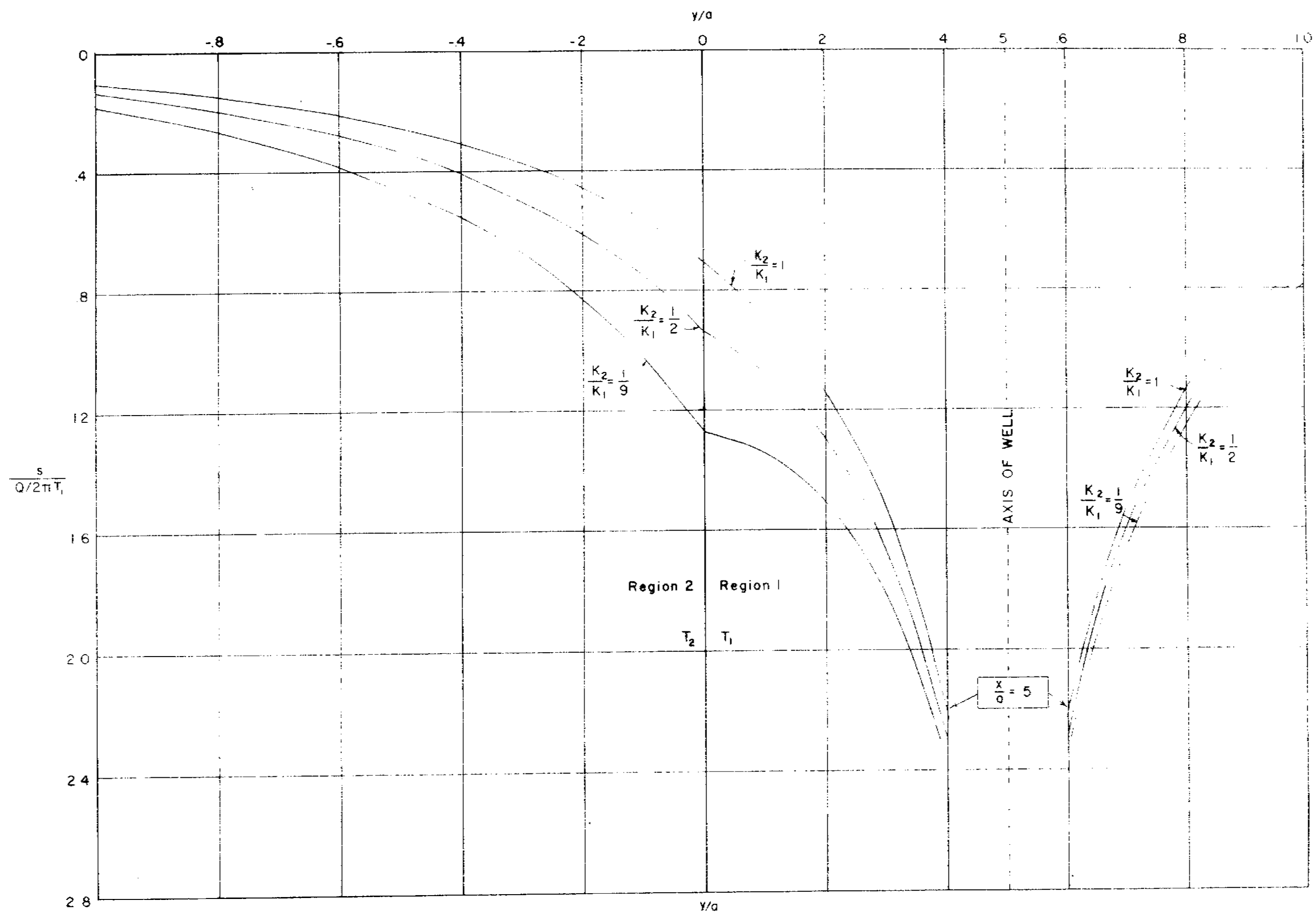


Fig. 10 -- Drawdown variation along a line parallel to the boundary and passing through the well for different values of δ.

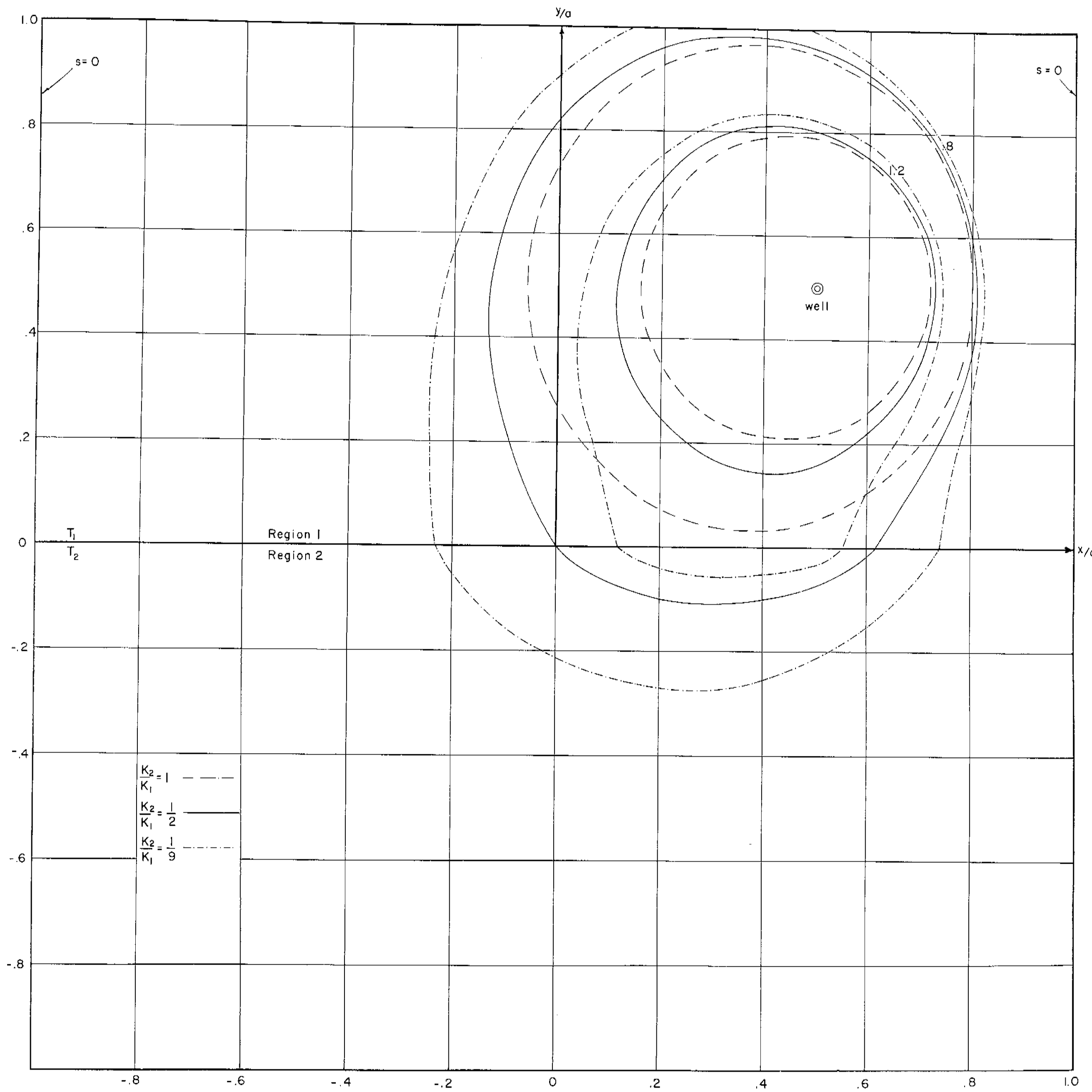


Fig. 11 -- Steady-state equi-drawdown curves (Problem 2, Case A) for different values of δ.

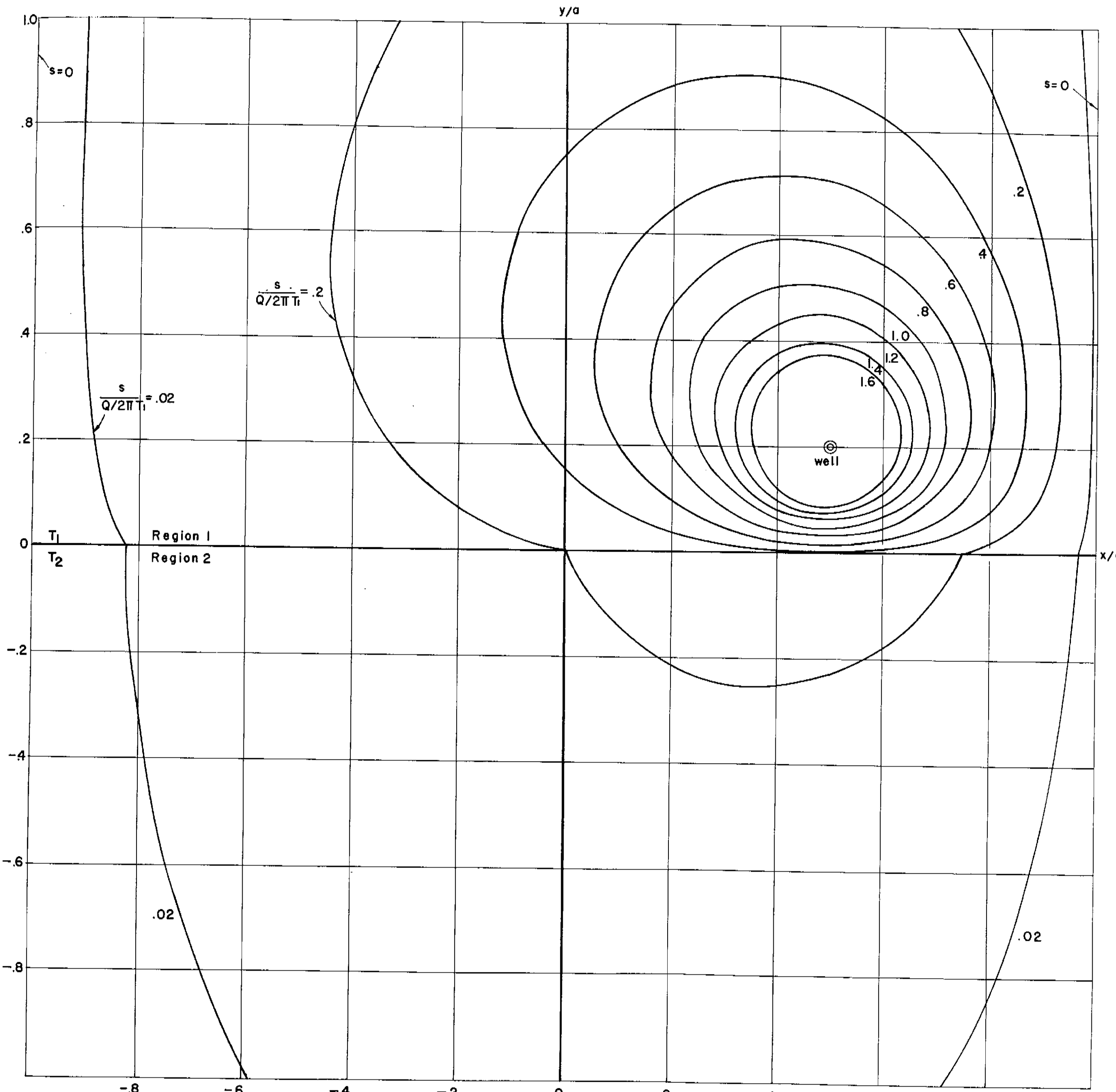


Fig. 12 -- Steady-state equi-drawdown lines of an infinite strip of aquifer having parallel boundaries maintained at zero drawdown, with well coordinates $x_1/a = 0.5$, $y_1/a = 0.2$ and $\delta = 10$.

Fig. 13 -- Cross-sections of figure 12 parallel to the x/a axis.

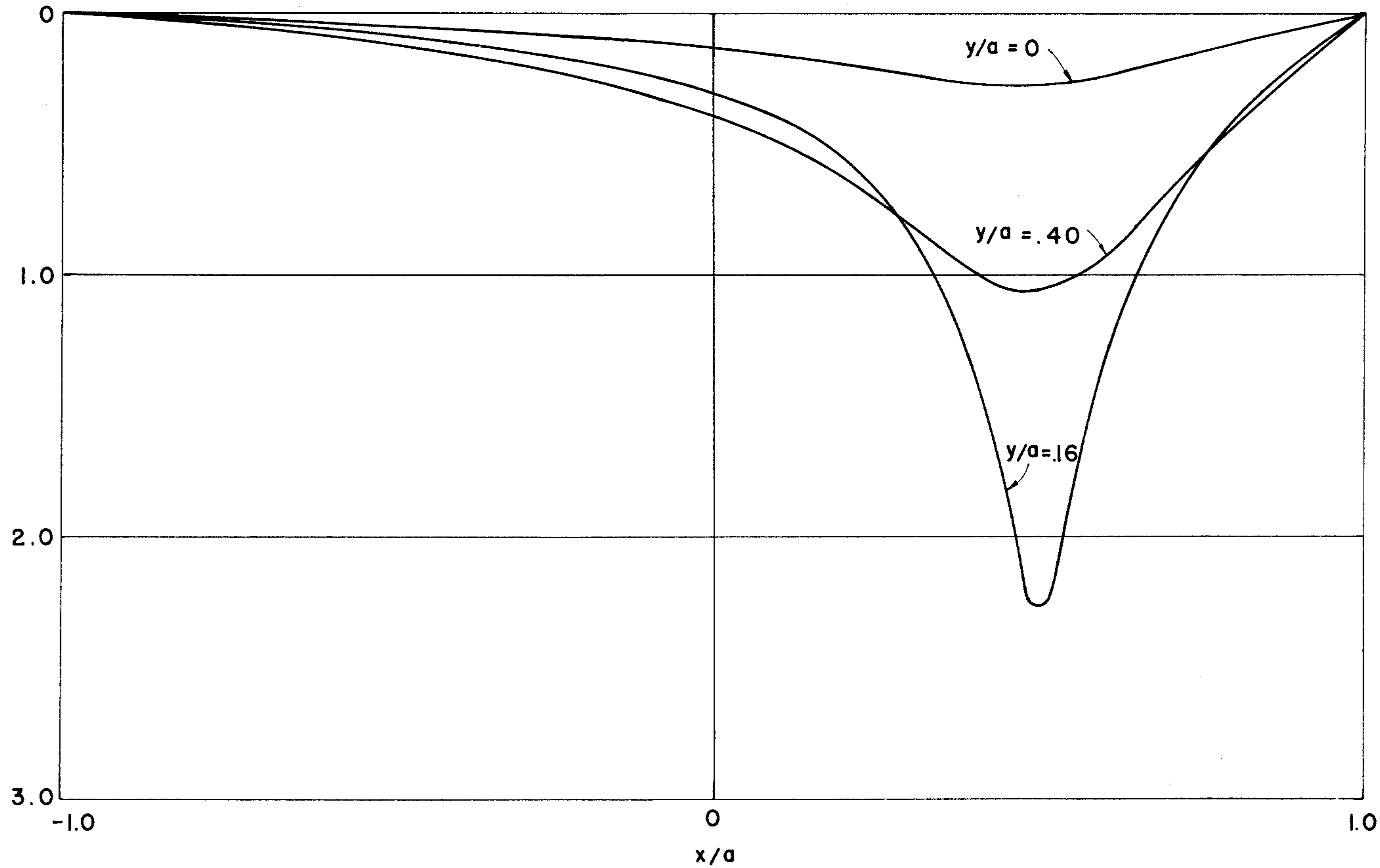
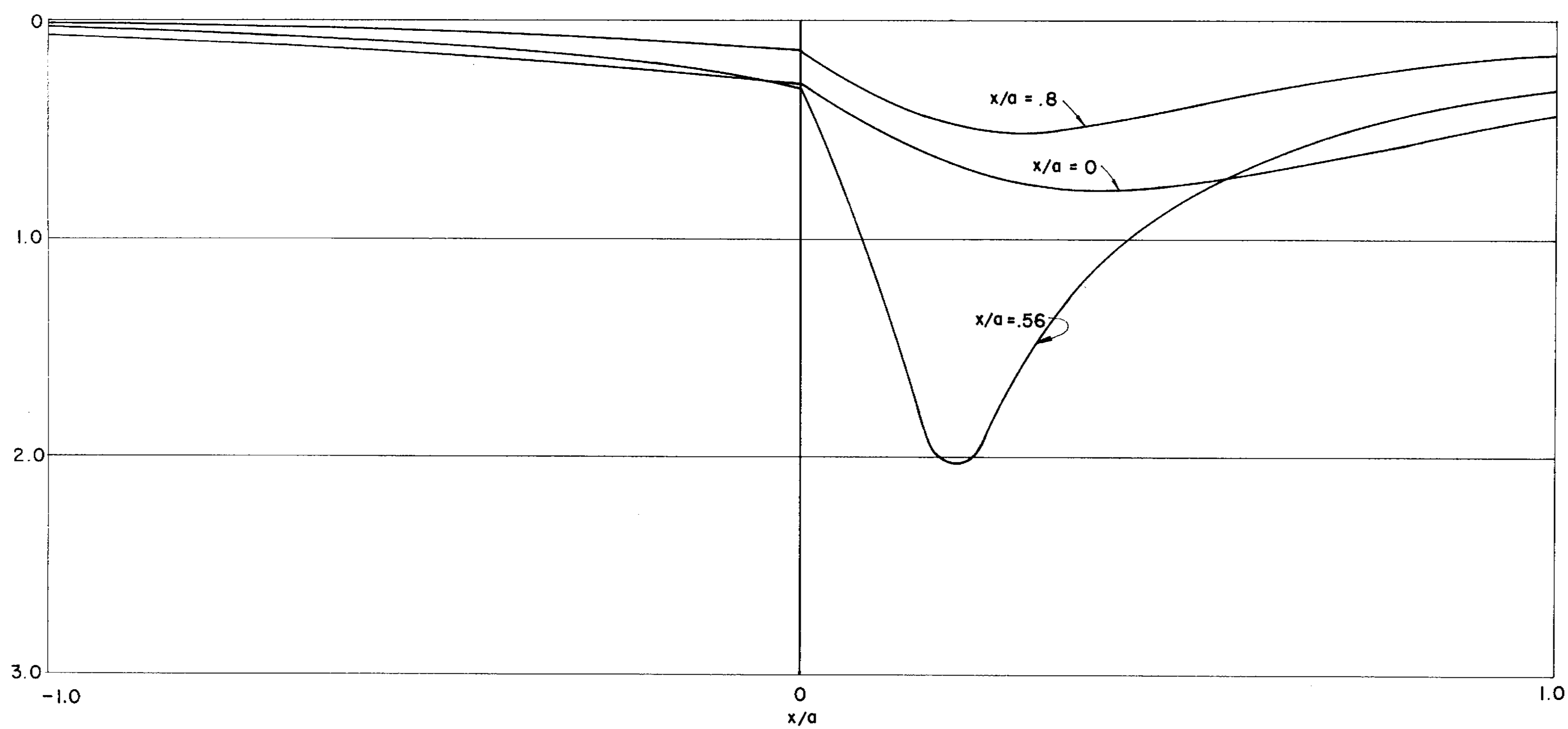


Fig. 14 -- Cross-sections of figure 12 parallel to the y/a axis.



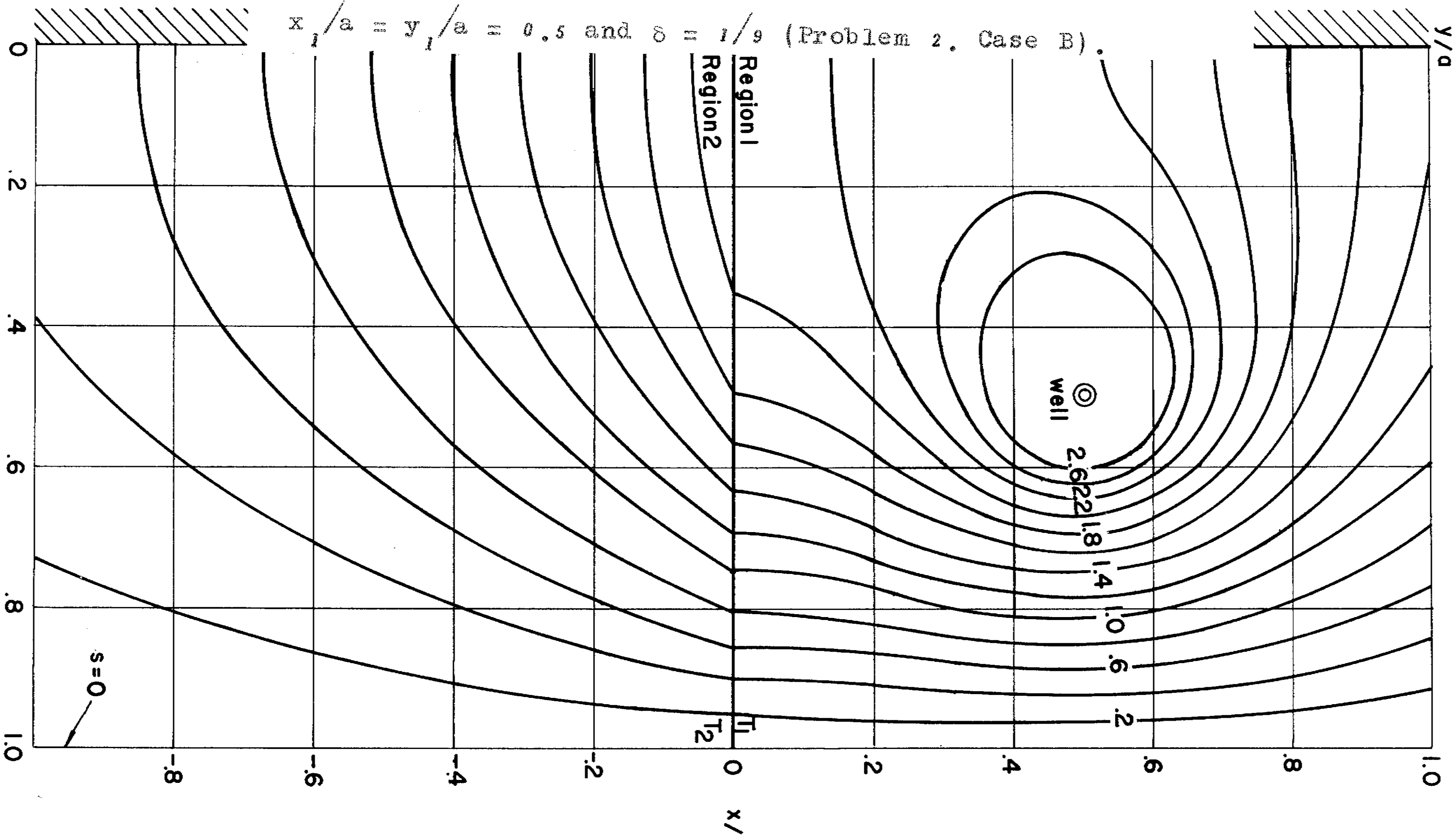
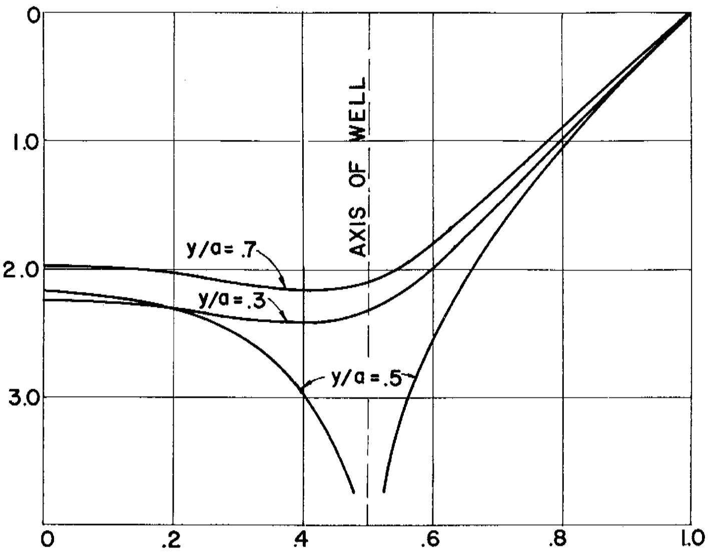


Fig. 15 -- Steady-state equi-drawdown lines of an infinite strip having one parallel boundary maintained at zero drawdown and the other at zero flux with well coordinates $x_1/a = y_1/a = 0.5$ and $\delta = 1/9$ (Problem 2, Case B).

Fig. 16 -- Cross-sections of figure 15 parallel to the x/a axis.



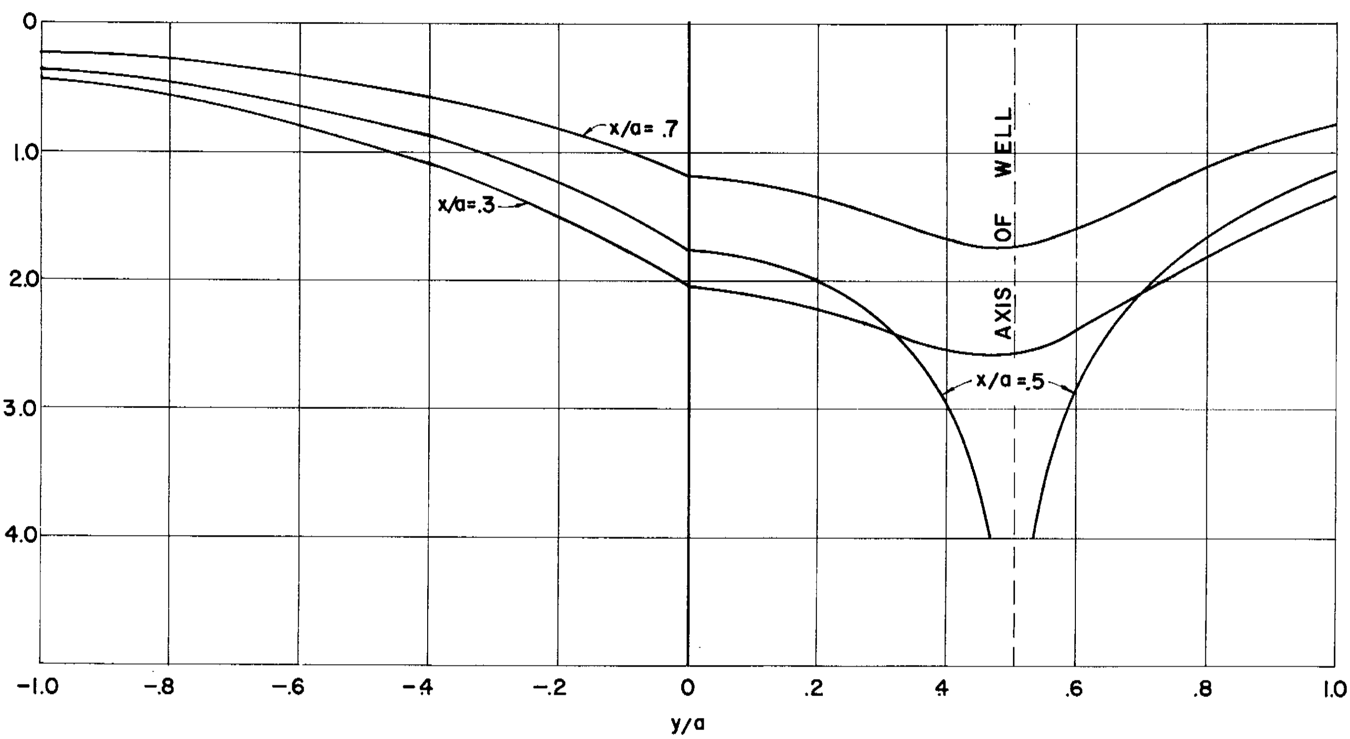


Fig. 17 -- Cross-sections of figure 15 parallel to the y/a axis.

Fig. 18 -- Steady-state equi-drawdown lines of an infinite strip illustrating the use of an "image well" to produce zero flux across a parallel boundary (Problem 2. Case B) with well coordinates $x_1/a = y_1/a = 0.5$ and $\delta = 1/2$.

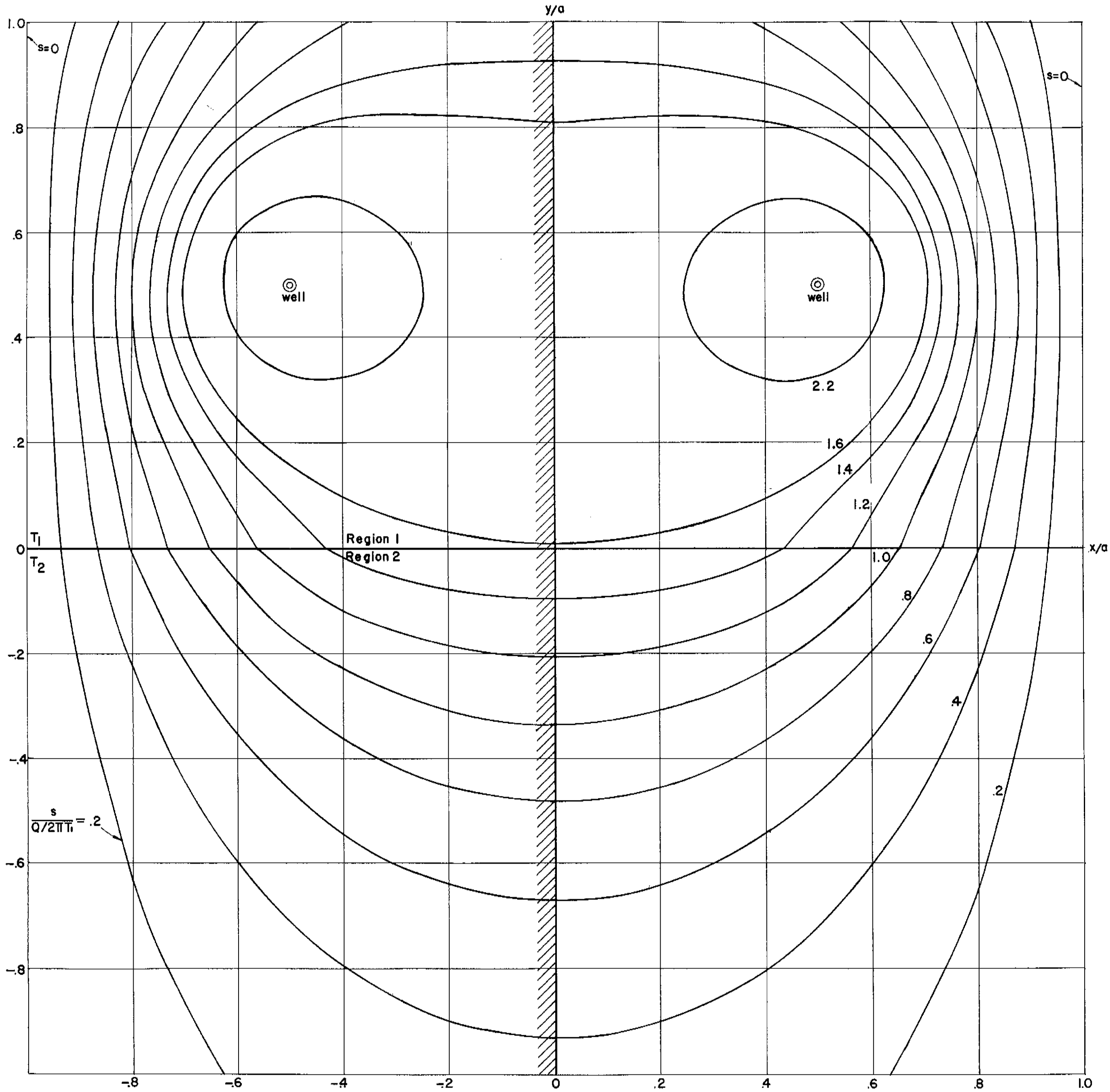
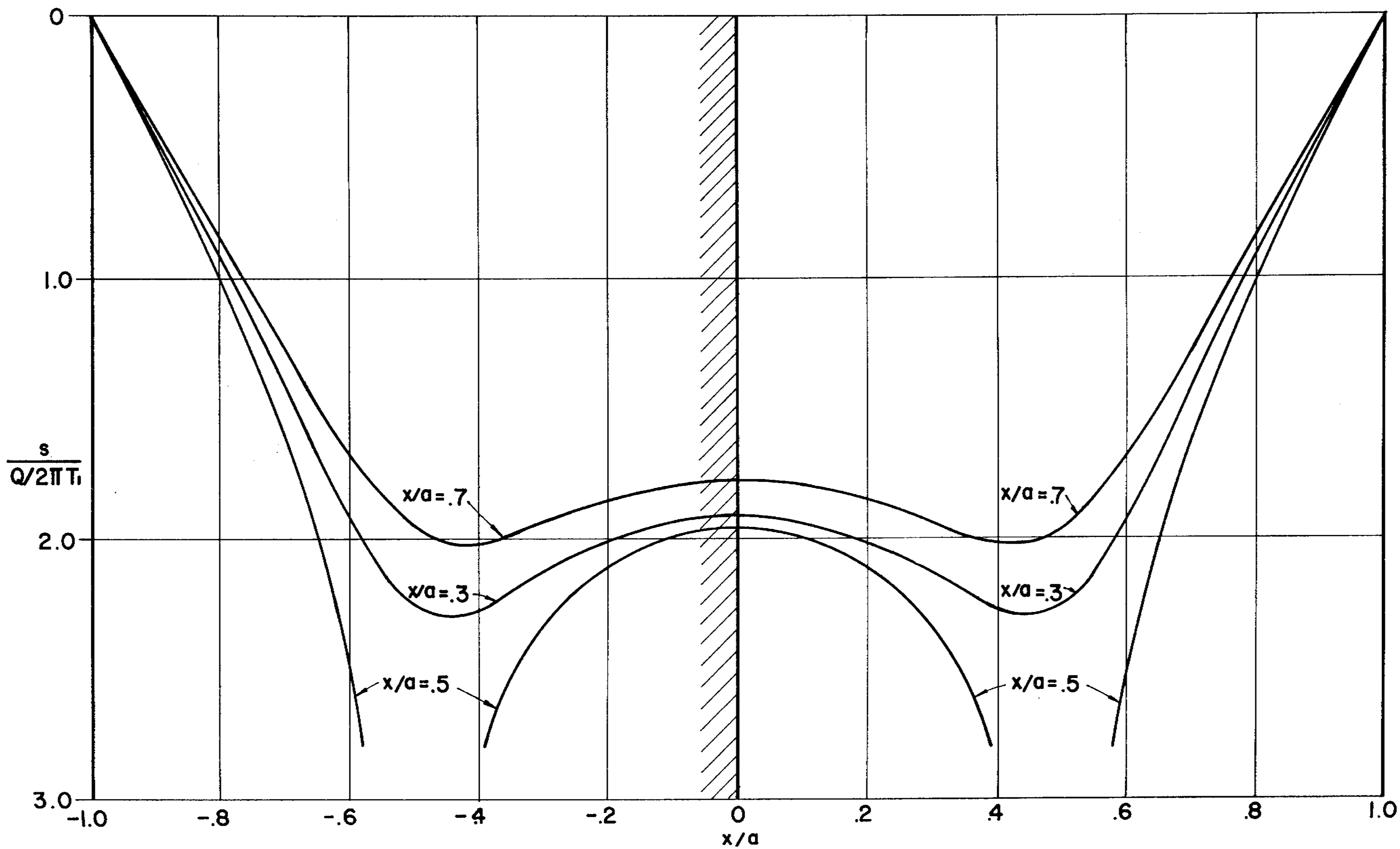


Fig. 19 -- Cross-sections of figure 18 parallel to the x/a axis.



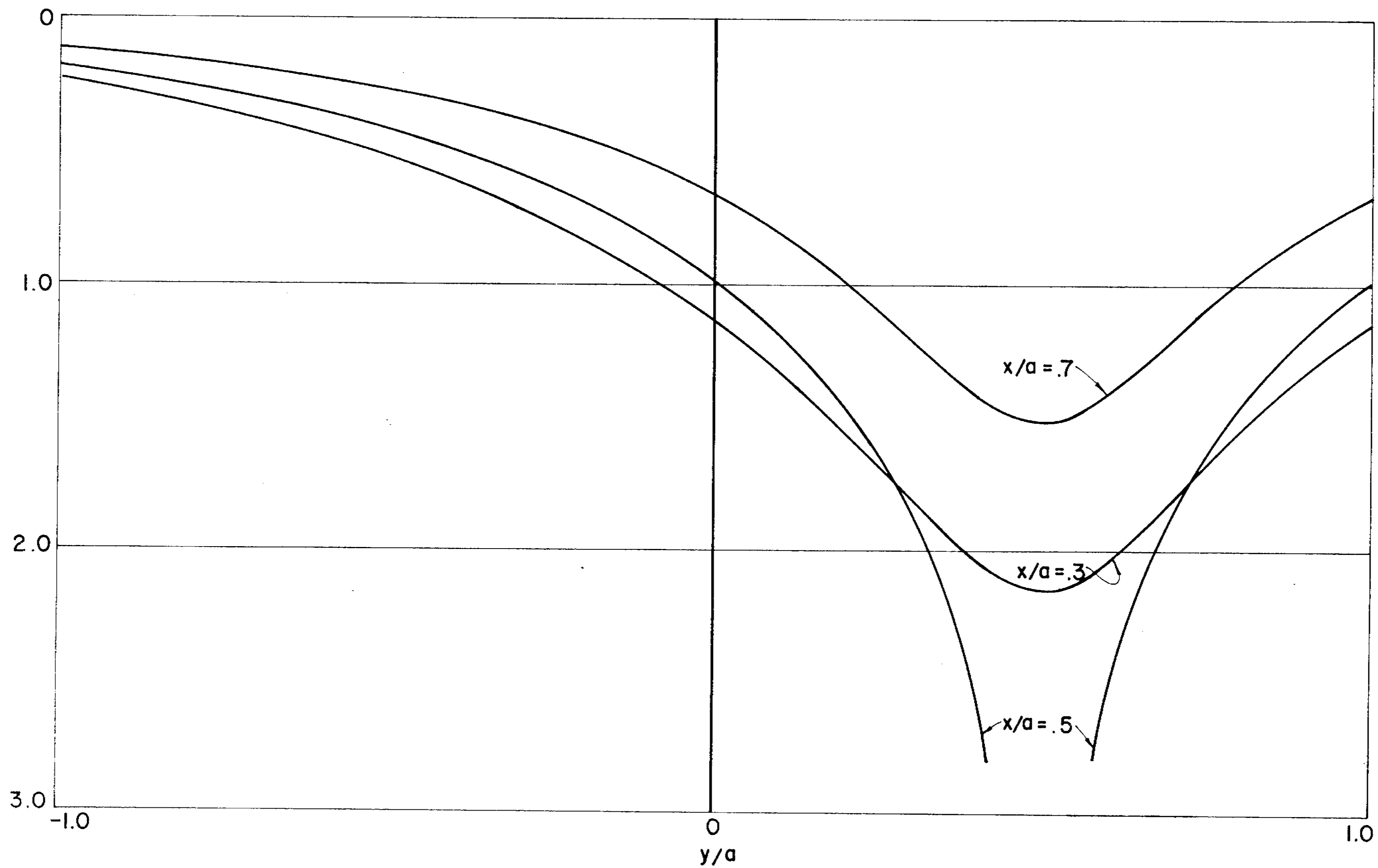
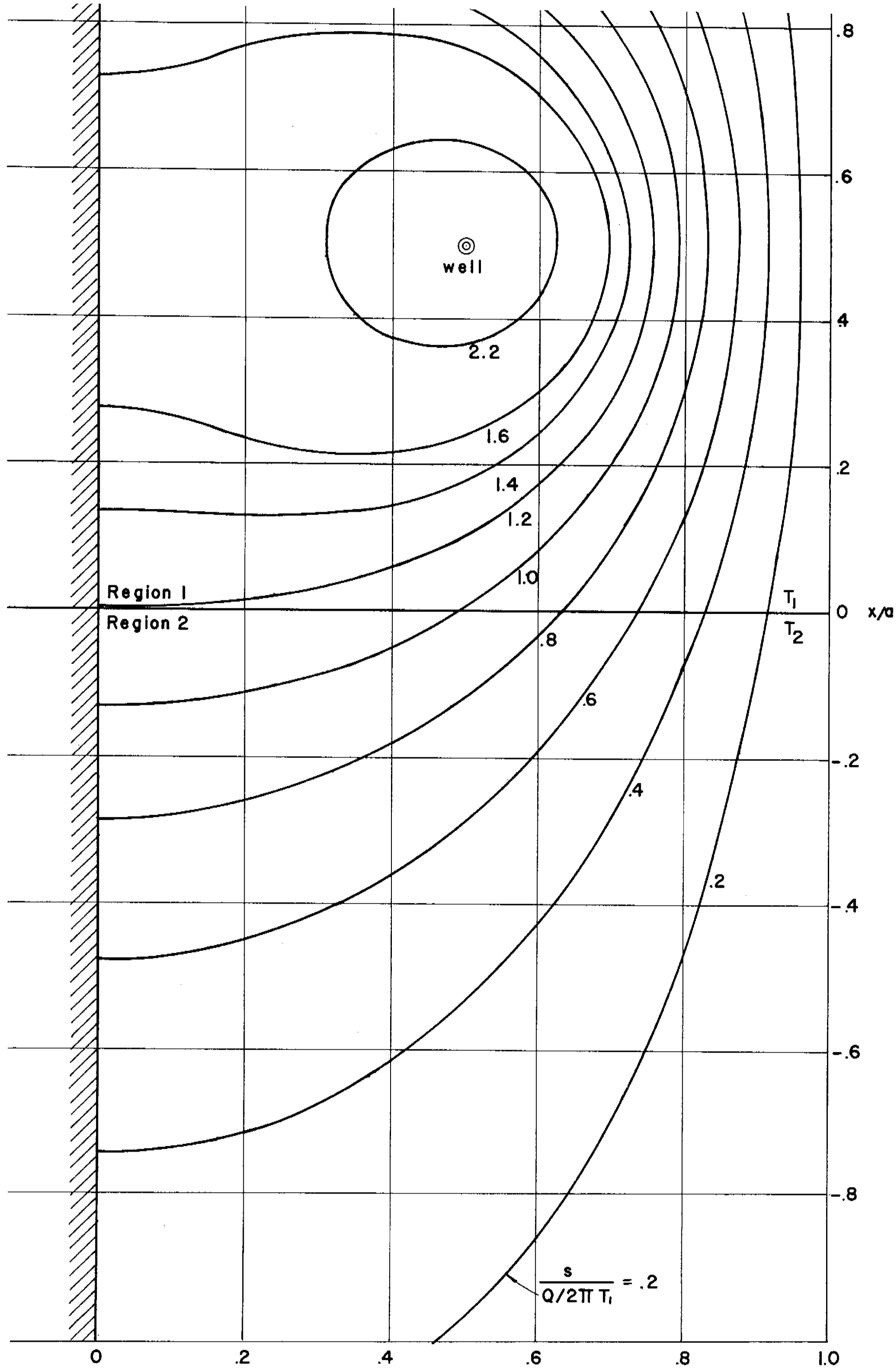


Fig. 20 -- Cross-sections of figure 18 parallel to the y/a axis.

Fig. 21 -- Steady-state equi-drawdown lines of an infinite strip having one parallel boundary maintained at zero drawdown and the other at zero flux with well coordinates $x_1/a = y_1/a = 0.5$ and $\delta = 1$ (Problem 2, Case B).



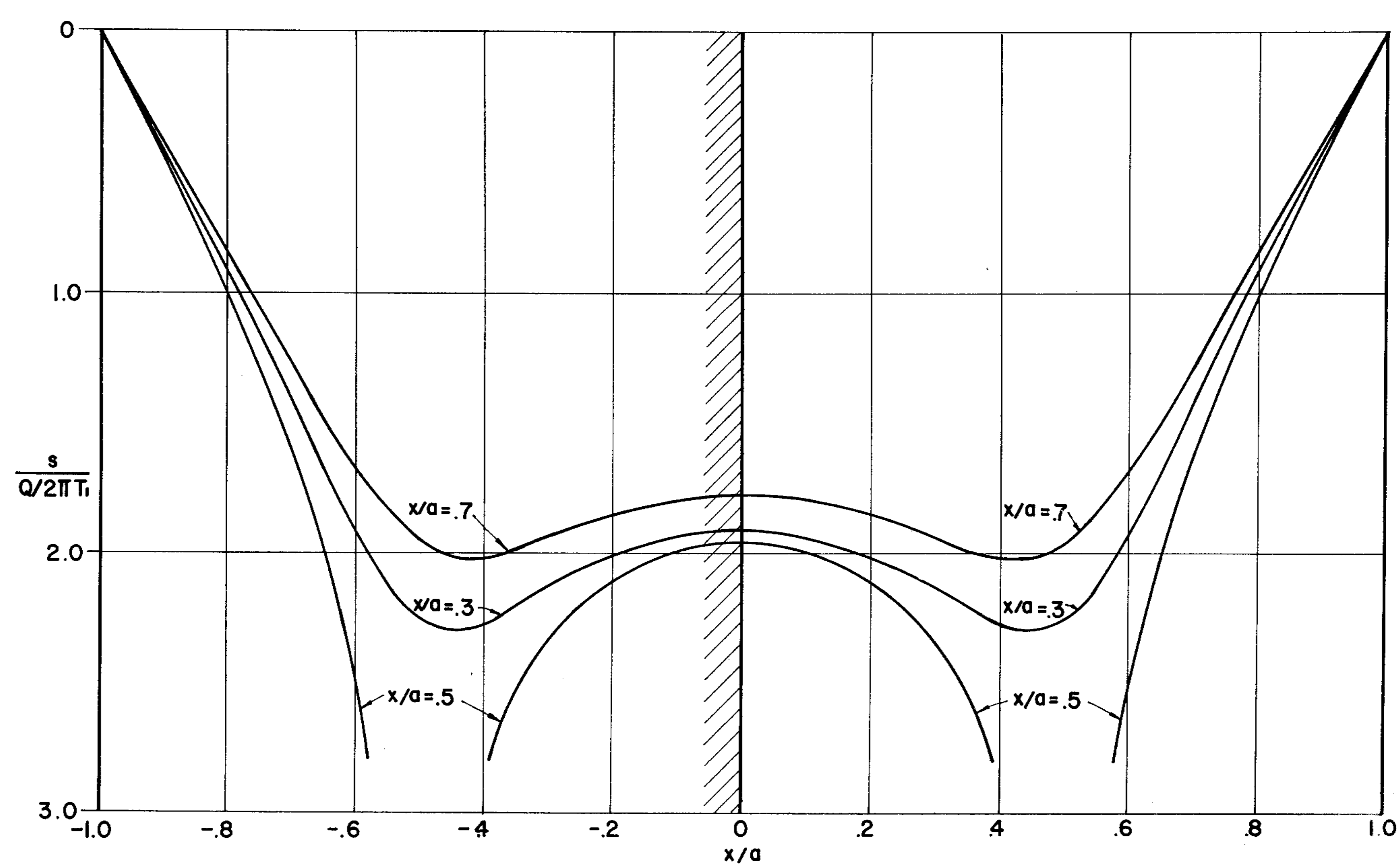
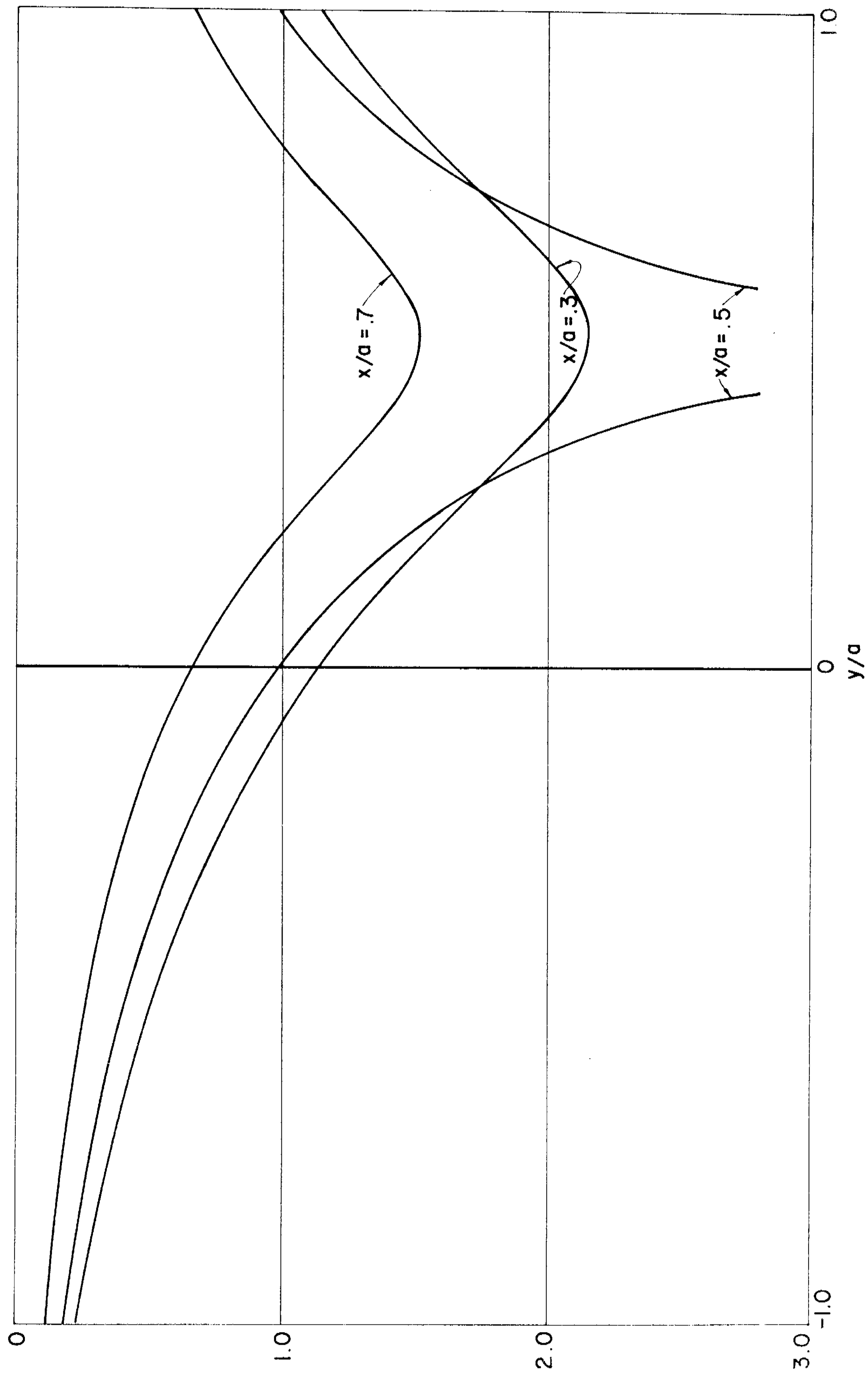


Fig. 22 -- Cross-sections of figure 21 parallel to the

Fig. 23 -- Cross-sections of figure 21 parallel to the y/a axis.



P A R T I V

SUMMARY AND CONCLUSIONS

SUMMARY

Mathematical models have been set up and solutions obtained for two flow systems. Each system is composed of two homogeneous, contiguous aquifers in a single horizontal layer of uniform thickness. The layer is bounded above and below by impermeable boundaries. Non-steady and when possible, steady-state solutions are presented.

The first problem considered is "flow toward a slit", (plane sink). Solutions are presented for two cases; "case A, constant drawdown at the slit", and "case B, constant discharge at the slit". These cases are represented diagrammatically by figure 1.

When the slit is maintained at constant drawdown (problem 1, case A) the general solution is given by equations (15) and (16). The infinite series in these equations are rapidly convergent for small values of time; however, for larger values of time an approximate solution is developed [equations (19) and (20)]. The solution at the interface between contiguous regions [equation (21)] is easily obtained from the general solution. Equation (22) represents the discharge of the slit. For larger values of time the discharge can be approximated by equation (23).

Solutions for three special cases are presented: a homogeneous region of large extent [equation (24)]; a region bounded by a slit, and an impermeable boundary [equation (25)], and a region bounded by a source of recharge of very high hydraulic conductivity and drained by a slit [equation (26)].

When the slit is maintained at constant discharge (problem 1, case B.) the general solution is given by equations (41) and (42). For larger values of time an approximate solution is developed [equations (45) and (46)]. Solutions for three special cases similar to problem 1, case A. are presented [equations (47), (48), and (49)].

The second problem considered is "Flow toward a steadily discharging well in an infinite strip". Solutions are presented for three cases: "case A, infinite strip composed of two contiguous aquifers whose boundaries are maintained at zero drawdown", "case B, zero drawdown on one boundary and zero flux across the other", and "case C, zero flux across both boundaries. These cases are represented diagrammatically by figure 2.

The general solution of problem 2, case A. is given by equations (67) and (68). The steady-state solution [equations (69) and (70)] is suitable for high speed computer tabulation. The results of such a tabulation are illustrated graphically (part III. "Solutions for special cases").

The general solution of problem 2, case B. is given by equations (72) and (73). Tabulation of the steady-state values, [equations (74) and (75)] are illustrated graphically in a manner similar to problem 1, case A.

The general solution of problem 2, case C, is given by equations (92) and (93). In this case there is no steady-state solution.

Specific conclusions

Mathematical expressions are developed which describe the non-steady drawdown distributions for both problems. The problem "flow toward a slit" has solutions in the form of infinite series which converge rapidly for small values of time. For larger values of time, approximate solutions greatly facilitate drawdown calculations. Solutions of practical interest can be obtained when the formation constants in the general solutions assume specific values.

The problem "flow toward a steadily discharging well in an infinite strip" has non-steady solutions which include terms in the form of definite and indefinite integrals. These integrals do not readily lend themselves to tabulation. The steady-state solutions, however, are suitable for calculations.

The solutions obtained are for artesian flow systems. However, they can be adapted for water-table flow systems if the maximum drawdown is small compared to the original depth of flow [Jacob, 1950].

Recommendations

Numerical evaluation and tabulation of the integrals V_1, V_2, W_1, and W_2 are necessary for application of the non-steady solutions of problem 2. High speed computer methods make such a tabulation possible.

A P P E N D I X

APPENDIX I

When not tabulated, the inverse transformation of a function can be determined by the method of integration in the complex plane [Churchill, 1958, p.176]. In general, this entails examining the function for singular points and branch cuts, then choosing a suitable path of integration that excludes these singularities. The inverse transformation is then equal to the line integral plus the sum of the residues due to the singularities. The inverse transformation of

$$f(p) = \frac{\exp[-a_1\sqrt{a+p} - b_1\sqrt{b+p}\,]}{p(\sqrt{a+p} + k\sqrt{b+p}\,)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (94)$$

where a_1, b_1, a, and b are constants, can be determined in this manner. The inversion integral can be written in the form

$$f(t) = \frac{1}{2\pi i}\int \frac{e^{zt}\exp[-a_1\sqrt{a+z} - b_1\sqrt{b+z}\,]\,dz}{z(\sqrt{a+z} + k\sqrt{b+z}\,)}$$

$$+ \text{(sum of the residues)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (95)$$

where the line integral is to be taken along the paths $\overline{AB}$, $\overline{BC}$, $\overline{CB}$, and $\overline{BA}$ (see fig. 24).

The integration is easily set up using the substitutions indicated in Table 5. After combining terms and simplifying, one can write the inverse transform in the form

$$f(t) = \frac{2e^{-at}}{\pi} \int_0^{\sqrt{a-b}} \exp(\beta^2 t - a_1\beta) \cdot$$

$$\frac{[\beta \sin(b_1\sqrt{a-b-\beta^2} + k\sqrt{a-b-\beta^2}) \cos(b_1\sqrt{a-b-\beta^2})] \beta \, d\beta}{(\beta^2-a)[\beta^2 + k^2(a-b-\beta^2)]}$$

$$- \frac{2e^{-at}}{\pi} \int_0^{\infty} \frac{e^{-\beta^2 t} \cos(a_1\beta + b_1\sqrt{a-b+\beta^2}) \beta \, d\beta}{(\beta^2+a)(\beta + k\sqrt{a-b+\beta^2})}$$

$$+ \frac{e^{-a_1\sqrt{a} - b_1\sqrt{b}}}{\sqrt{a} + k\sqrt{b}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (96)$$

where the last term of (96) is the residue at the simple pole $p = 0$. The integration around the circles of centers at $p = -a$ and $p = -b$ approaches zero as the radii approaches zero.

The second integral in equation (96) can be simplified when it is of the form

$$g(y) = \int_0^\infty F(x)\cos(xy)\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (97)$$

where $F(x) = \frac{e^{-c^2x^2}}{v^2+x^2}$, $c^2 > 0$, $v > 0$ [Bateman, 1954, p. 15]

and

$$g(y) = \frac{\pi}{4v} e^{c^2v^2} \left[e^{-vy}\operatorname{erfc}(cv - \tfrac{1}{2}c^{-1}y) + e^{vy}\operatorname{erfc}(cv + \tfrac{1}{2}c^{-1}y)\right]$$

$c > 0$.

In problem 2, case A, this relation was used to obtain the final solution. As a check, the method of convolution may be used to find the inverse transform of

$$f(p) = \frac{\exp[-a\sqrt{a+p}\ ]}{p\sqrt{a+p}}$$

The results of the two methods are identical, provided one uses the equation

$$\frac{2}{\sqrt{\pi}} \int_0^X \exp[-x^2 - (\frac{y}{x})^2] \, dx =$$

$$\frac{1}{2} [\exp(-2y)\mathrm{erfc}(\frac{y}{X} - X) - \exp(2y)\mathrm{erfc}(\frac{y}{X} + X)] \quad \ldots . (98)$$

After the terms are combined, the results of the transformation are simplified further by using the relation

$$\sum_{n=1}^{\infty} \frac{e^{-\frac{n\pi}{2a}(y \pm y_1)}}{n} R_n = \frac{1}{2} \ln \frac{\cosh\frac{\pi}{2a}(y \pm y_1) + \cos\frac{\pi}{2a}(x+x_1)}{\cosh\frac{\pi}{2a}(y \pm y_1) - \cos\frac{\pi}{2a}(x-x_1)} \quad \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (99)$$

TABLE 5

ANALYSIS OF INTEGRATION PATHS SHOWN IN FIGURE (27)

$(a + z = u_1^2 e^{i\Theta_1}, \; b + z = u_2^2 e^{i\Theta_2})$

Path $\overline{AB}$, $\Theta_1 = -\pi$, $\Theta_2 = -\pi$, $u_2^2 - u_1^2 = a-b$

$$\sqrt{a + z} = -iu_1$$

$$\sqrt{b + z} = -iu_2$$

$$z = -u_1^2 - a$$

$$dz = -2u_1 du_1$$

Upper limit: When $z = -a$, $u_1 = 0$.

Lower limit: When $z = -\infty$, $u_1 = \infty$.

Path $\overline{BC}$, $\Theta_1 = 0$, $\Theta_2 = -\pi$, $u_2^2 + u_1^2 = a - b$

$$\sqrt{a + z} = u_1$$

$$\sqrt{b + z} = -iu_2$$

$$z = u_1^2 - a$$

$$dz = 2u_1 du_1$$

Upper limit: When $z = -b$, $u_1 = \sqrt{a-b}$.

Lower limit: When $z = -a$, $u_1 = 0$.

Path $\overline{CB}$, $\Theta_1 = 0$, $\Theta_2 = \pi$, $u_2^2 + u_1^2 = a - b$

$$\sqrt{a + z} = u_1$$

$$\sqrt{b + z} = iu_2$$

$$z = u_1^2 - a$$

$$dz = 2u_1 du_1$$

Upper limit: When $z = -a$, $u_1 = 0$.

Lower limit: When $z - b$, $u_1 = \sqrt{a - b}$.

Path $\overline{BA}$, $\Theta_1 = \pi$, $\Theta_2 = \pi$, $u_2^2 - u_1^2 = a - b$

$$\sqrt{a + z} = iu_1$$

$$\sqrt{b + z} = iu_2$$

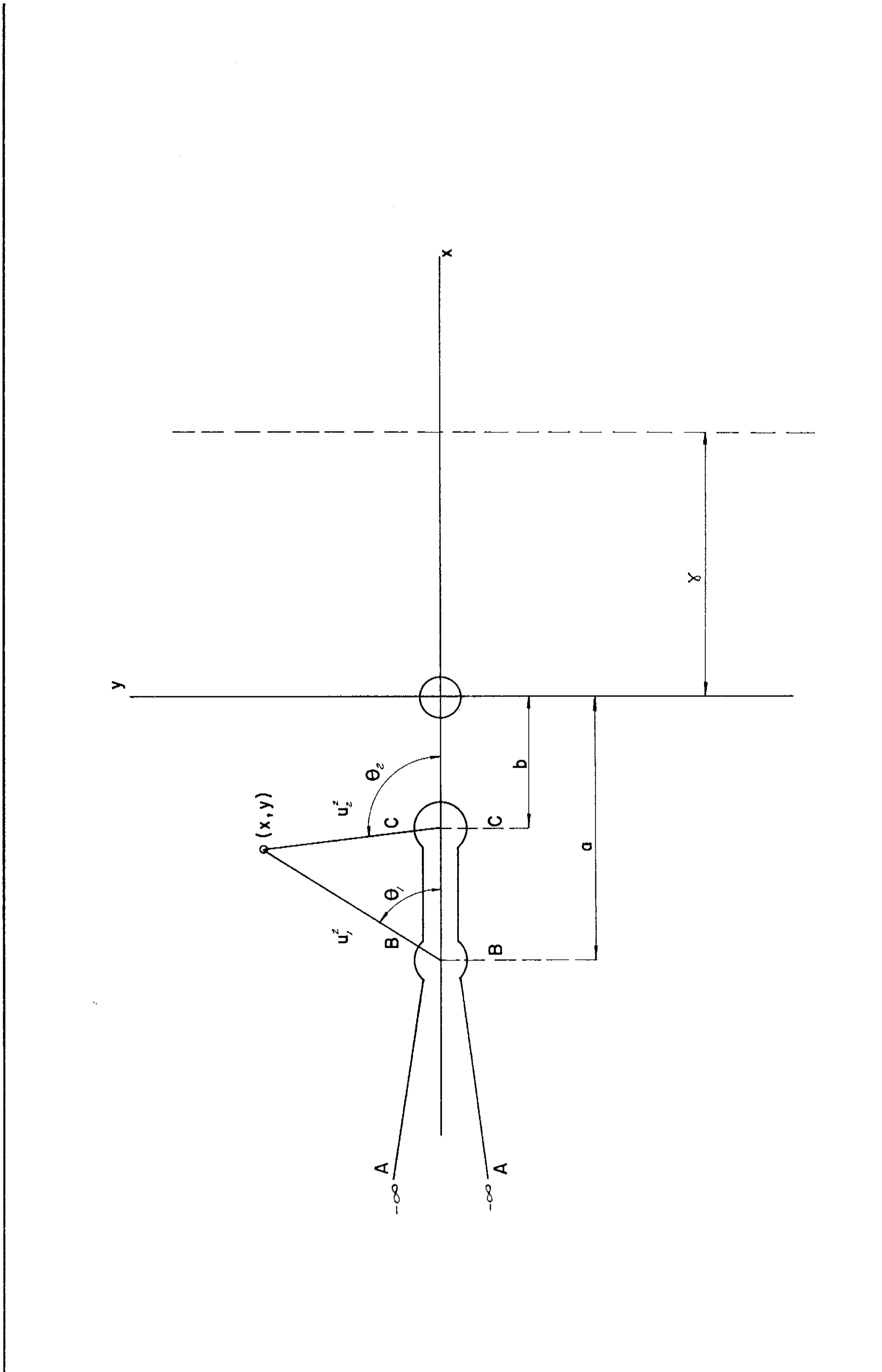
$$z = -u_1^2 - a$$

$$dz = -2u_1 du_1$$

Upper limit: When $z = -\infty$, $u_1 = \infty$.

Lower limit: When $z = -a$, $u_1 = \infty$.

Fig. 24 -- Integration contour of the inversion integral (equation 95).



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