NONSTEADY FLOW TOWARD WELLS
WHICH PARTIALLY PENETRATE
THICK ARTESIAN AQUIFERS

by

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The nonsteady drawdown distribution for wells partially penetrating and nonpenetrating an artesian aquifer of infinite areal extent and unlimited thickness is solved. The confined aquifer is of uniform hydraulic conductivity and uniform specific storage. Two flow systems are considered: (1) wells just tapping an artesian aquifer and (2) wells penetrating an artesian aquifer. The solutions are obtained by the method of superposition using the solution of a spherical cavity located at \((0, z_1)\); this basic solution is:

\[
S_c = \frac{Q}{4\pi K} \left[ \frac{\text{erfc} \left( \frac{r^2 + (z - z_1)^2}{4\alpha t} \right)}{r^2 + (z - z_1)^2} + \frac{\text{erfc} \left( \frac{r^2 + (z + z_1)^2}{4\alpha t} \right)}{r^2 + (z + z_1)^2} \right]
\]

The variation of drawdown with time, and distance, for aquifer of unlimited thickness is compared with known solution for aquifer of limited thickness by means of graphs.

The function \(F(u, x) = \int_{u}^{\infty} e^{-\beta} \text{erf}(u\sqrt{\beta}) d\beta\), involved in the mathematical solutions is evaluated numerically (table 1). Graphical methods based on the derived solutions are outlined for determining the formation constants. These methods are: (1) The
type-curve method, (2) the inflection-point method, and (3) the
two-point method. Applications of the type-curve method are
illustrated by treating data from Luna and Quay Counties, New
Mexico.
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INTRODUCTION

General

Frequently, producing wells do not penetrate completely the aquifer from which they are pumping. The hydraulics of such wells is therefore different from wells which fully penetrate the aquifer.

The problem of partial penetration has long been recognized, and approximate solutions for various field conditions have been advanced. Most writers (see list of references) have assumed that the ground-water flow toward the well is stabilized; in other words, that a steady-state condition is attained. Such a condition rarely obtains during periods of actual well operation.

When dealing with nonsteady-flow problems of partial penetration, the current practice is to make use of the nonsteady-flow formulas for complete penetration either with or without adjustments. These adjustments, if used, are based on the steady-state flow formulas for partial penetration, depending on the presumably known characteristics of the aquifer. For example, if it is desired to compute the drawdown values in areas fairly close to the well, and if the horizontal permeability of the aquifer is very high, the
bottom of the partially penetrated aquifer often is assumed to coincide with the bottom of the pumping well; that is, the flow condition is assumed to be one of complete penetration, the depth of the well equaling the thickness of the aquifer. It has also been customary to use the formulas of complete penetration to describe the flow in cases of partial penetration when (1) the drawdown observations are made in observation holes that completely penetrate the aquifer, (2) the drawdown values are required or observed at distances which are far away from the pumping well (that is, at distances such that the flow in the aquifer is more or less purely radial), or (3) the average of the drawdowns observed at the top and at the bottom of the aquifer is used in determining the formation coefficients.

Approximating the flow problems by procedures such as those mentioned above may give fair results in the case of an aquifer of relatively small thickness, provided that the conditions assumed in each case obtain. In the case, however, of extensive aquifers (aquifers of very great thickness), such procedures are not dependable. Either they may give erroneous results because of the approximate nature of the assumptions involved, or they may not be practical to carry out, or both.
Formulas for nonsteady flow toward a steady well partially penetrating an infinite artesian aquifer of limited thickness, and for nonsteady flow toward a well partially penetrating a thick water-table aquifer, have been developed by Hantush (1957) and Boulton (1954) respectively. Because of the nature of the assumptions made by Boulton, his formula cannot be used in the case of confined (artesian) flow; that of Hantush, on the other hand, describes the flow in leaky and nonleaky aquifers of both limited and unlimited thicknesses. The Hantush partial penetration formula can also be used, under certain conditions, to describe the flow in arrested wells that are drawing a thick water-table aquifer of infinite free aquifers. Computations using this formula in its present forms, although easily performed for aquifers of relatively small thicknesses, become lengthy and tedious in the case of very thick aquifers. The present approach is used throughout the paper, aquifers unless the infinite integral appearing in the second form of the solution is tabulated.

**Purpose**

The purpose of the present work is threefold:

1. To obtain solutions, for different flow conditions, for the drawdown distribution around a steady well that partially penetrates an infinite elastic artesian aquifer of unlimited depth.
2. To determine the degree of penetration in an aquifer of limited thickness, below which the flow pattern can be computed by the formula obtained for aquifers of unlimited depth.

3. To describe methods by which the formation constants can be obtained by aquifer pumping tests, when the appropriate solutions are used.

Statement of Problem

The problem is to determine the drawdown distribution around wells that are draining a thick artesian aquifer of infinite areal extent. It is assumed that the hydraulic conductivity and the specific storage remain constant both in space and time. Initially, the drawdown distribution is zero throughout the aquifer.

Two flow systems are considered:

Wells just tapping an artesian aquifer. In this system the water is discharged through a semispherical cavity at the bottom of the well, the radius of which is equal to that of the well bore. In practice such a flow system may represent flow toward: (a) wells of zero penetration and of constant discharge (fig. 1) or (b) artesian springs or wells of zero penetration and of constant head (fig. 2).
Wells penetrating an artesian aquifer. In this system the water is discharged through a cylinder whose length is equal to the depth of penetration. In considering this type of flow, two cases may arise in practice, that of flow toward a steadily discharging well, and that of a flowing well (well of constant drawdown).

Only the first case is treated in this thesis (fig. 3).
THEORY

Major Symbols and Definitions

$r$  Radial distance from the axis of the pumping well to any point in the space.  \( (L) \)

$z$  Depth of any point below the surface of the impermeable layer.  \( (L) \)

$z_1$  Location of a spherical cavity at \( r = 0 \).  \( (L) \)

$R_1 = \sqrt{r^2 + z^2}$, the distance from the origin of the coordinate system to any point in space.  \( (L) \)

$R = \sqrt{r^2 + (z - z_1)^2}$, the distance from a point located at \( z_1 \) to any point in space.  \( (L) \)

$R_w$  Effective radius of the spherical cavity.  \( (L) \)

$r_w$  Effective radius of the well.  \( (L) \)

$t$  Time since pumping started.  \( (T) \)

$t_1$  Time at which the inflection point takes place.  \( (T) \)
\( f(p) \quad \text{Laplace transform, or the image, of } f(t). \)

\( b \quad \text{Depth of the pumping well below the surface of the impermeable layer. (L)} \)

\( b' \quad \text{Length of the perforations in observation wells. (L)} \)

\( s \quad \text{Drawdown at any point } (r, z) \text{ in the aquifer at any time } t \text{ since startup. (L)} \)

\( s_c \quad \text{Drawdown at any point } (r, z, t) \text{ in the aquifer due to a spherical cavity at the point } (0, z_1). \)

\( s_a \quad \text{Average drawdown in an observation well of perforated length } b'. \quad (L) \)

\( s_m \quad \text{Maximum drawdown at any point } (r, z). \quad (L) \)

\( s_{am} \quad \text{Maximum average drawdown in an observation well of perforated length } b'. \quad (L) \)

\( s_w \quad \text{Drawdown at the pumping well at any time.} \quad (L) \)

\( s_i \quad \text{Drawdown at the inflection point.} \quad (L) \)

\( Q \quad \text{Rate of discharge of the well during pumping. } (L^3/T) \)
K  Hydraulic conductivity of the aquifer. \((L/T)\)

\(S\)  Specific storage, defined as the amount of water which

a unit volume of the aquifer releases from storage under

a unit head decline. \((L^{-1})\)

\(\alpha\)  Ratio \(\frac{K}{S}\). \((L^2/T)\)

\(u\)  Relation \(\frac{r^2 S}{4Kt}\).

\(\text{erf}(x)\)  Error function \(= \frac{2}{\sqrt{\pi}} \int_0^x e^{-\beta^2} d\beta\).

\(\text{erfc}(x)\)  Complementary error function \(= \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\beta^2} d\beta\).

\(= 1 - \text{erf}(x)\)
Fig-1 Well just tapping the aquifer case 1 - with constant discharge

Fig-2 Well just tapping the aquifer case 2 - with constant drawdown

Fig-3 Well penetrating the aquifer

Fig-4 Flow toward a spherical cavity

Fig-5 Image system of the spherical cavity
Differential Equation of Motion

The general differential equation of ground-water motion is

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{\partial s}{\partial t}$$

In the cylindrical coordinates, the equation is

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial r^2} + \frac{\partial^2 s}{r \partial \theta^2} + \frac{\partial^2 s}{\partial z^2} = \frac{\partial s}{\partial t}$$

For purely spherical flow, the equation reduces to

$$\frac{\partial^2 s}{\partial R^2} + \frac{2 \partial s}{R \partial R} = \frac{\partial s}{\partial t}$$

For the derivation of these equations, the reader is referred to Jacob (1940), Muskat (1937), and others (see list of references).
Although several procedures can be employed in solving the problems treated below, the one followed here is believed to be the simplest and the most direct. The method of Laplace transformation is used to obtain a solution for the drawdown around a spherical cavity (fig. 4) draining an aquifer of infinite depth and of infinite areal extent. From this, the solutions of the different cases treated here are built up by the method of superposition.

**Flow toward a spherical cavity**

The system of flow being spherical, with the origin in this case taken at the center of the cavity \( (0, z_1) \), the boundary-value problem is:

\[
\frac{\partial^2 s}{\partial R^2} + \frac{2\partial s}{R \partial R} - \frac{\partial s}{\partial t} = \alpha \frac{\partial s}{\partial t} \tag{1}
\]

\[
s (R, 0) = 0 \tag{2}
\]

\[
s (\infty, t) = 0 \tag{3}
\]

\[
\frac{\partial s}{\partial z} (r, 0, t) = 0 \tag{4}
\]

where \( R = \sqrt{r^2 + (z - z_1)^2} \), and \( \alpha = \frac{K}{S} \).
By using the Laplace transformation with respect to \( t \)
and applying boundary condition (2), the transformed boundary-
value problem in \((R, \varphi)\) will be:

\[
\frac{\partial^2 \bar{e}}{\partial R^2} + \frac{2 \partial \bar{e}}{R \partial R} = \frac{p\varphi}{\alpha}
\]

(5)

The integration around the coordinate \( \varphi \) will give

\[
\bar{e}(\infty, \varphi) = 0
\]

(6)

and

\[
\frac{\partial \bar{e}}{\partial \varphi}(x, 0, \varphi) = 0
\]

(7)

Further, assuming the momentum equation or given as equation

Equation (5) is reducible to the modified Bessel equation by the
substitution of \( \bar{e} = R^{-1/2} y \). After this substitution, equation (5)
assumes the form

\[
\frac{\partial^2 y}{\partial R^2} + \frac{\partial y}{R \partial R} - \left( \frac{1}{4R^2} + \frac{p}{\alpha} \right) y = 0
\]

(8)

Conditions (5), (6), and (7) are satisfied by

\[
\bar{e}_c = c_1 \left[ \frac{K_{1/2} \left( \sqrt{R^2 + (z - z_1)^2} \cdot \sqrt{\alpha} \right)}{\left[ R^2 + (z - z_1)^2 \right]^{1/4}} + \frac{K_{1/2} \left( \sqrt{R^2 + (z + z_1)^2} \cdot \sqrt{\alpha} \right)}{\left[ R^2 + (z + z_1)^2 \right]^{1/4}} \right]
\]

(9)

where \( K_{1/2} \) is the half-ordered modified Bessel function of the
second kind.
By using the relation (Watson, 1944) \( K_{1/2}(z) = \sqrt{\frac{\pi}{2\alpha}} \cdot e^{-z} \), equation (9) can be put in the form

\[
\frac{\dot{s}}{c} = c \left[ \exp \left[ -\sqrt{\frac{\alpha}{2}} \left\{ \frac{z^2}{z^2 + (z - z_1)^2} \right\} \right] \frac{\exp \left[ -\sqrt{\frac{\alpha}{2}} \left\{ \frac{r^2}{r^2 + (z + z_1)^2} \right\} \right]}{\sqrt{z^2 + (z - z_1)^2}} + \frac{\exp \left[ -\sqrt{\frac{\alpha}{2}} \left\{ \frac{r^2}{r^2 + (z + z_1)^2} \right\} \right]}{\sqrt{z^2 + (z + z_1)^2}} \right] \tag{10}
\]

The proper distribution around the spherical cavity will depend on the value of the constant \( c \) appearing in equation (10), which in turn depends on the boundary conditions of the case to be considered.

Having obtained the elementary solution as given by equation (10), the solutions to the different cases follow.

Spherical Cavity

The boundary-value problem of this case is given by equations (1), (2), (3), (4), and the following condition:

\[
4\pi KR^2 \frac{\partial \dot{s}}{\partial R} \bigg|_{R=0} = -Q \tag{11}
\]

By applying the Laplace transformation on (11), the transformed boundary condition in \((R, p)\) will be:
\[ 4\pi KR^2 \frac{d^2 \delta}{dR^2} = -\frac{Q}{p} \tag{12} \]

\[ R \rightarrow 0 \]

The value of the constant \( c \) is obtained by applying condition (12) to equation (10), which gives \( c = \frac{Q}{4\pi Kp} \). Thus the solution can finally be written as

\[ \bar{s} = \frac{Q}{4\pi Kp} \left[ \frac{\exp \left(-\sqrt{\frac{p}{\alpha}} \left\{ r^2 + (z-z_1)^2 \right\} \right)}{\sqrt{r^2 + (z-z_1)^2}} + \frac{\exp \left(-\sqrt{\frac{p}{\alpha}} \left\{ r^2 + (z+z_1)^2 \right\} \right)}{\sqrt{r^2 + (z+z_1)^2}} \right] \tag{13} \]

(a) Nonsteady solution

By using tables of inverse Laplace transformation (Churchill, 1958), the drawdown distribution will be given by

\[ s_c = \frac{Q}{4\pi K} \left[ \frac{\text{erfc} \sqrt{\frac{r^2 + (z-z_1)^2}{4\alpha t}}}{\sqrt{r^2 + (z-z_1)^2}} + \frac{\text{erfc} \sqrt{\frac{r^2 + (z+z_1)^2}{4\alpha t}}}{\sqrt{r^2 + (z+z_1)^2}} \right] \tag{14} \]

Average drawdown in wells of perforated depth: Equation (14) gives the drawdown at any point in the aquifer. The drawdown observed in wells of perforated section \( b' \) is the average value of the drawdowns at each point of the perforated length. Thus to obtain the average drawdown in a bore hole of length \( b' \), equation (14) is integrated with respect
to $z$ between the limits $0$ and $b'$, and divided by $b'$. The average drawdown $s_{ca}$ can be obtained as follows:

$$s_{ca} = \frac{Q}{4\pi Kb'} \left[ b' \left( \begin{array}{c} \text{erfc} \sqrt{\frac{r^2 + (z - z_1)^2}{4\alpha t}} \frac{r^2 + (z + z_1)^2}{4\alpha t} \\ \sqrt{r^2 + (z - z_1)^2} \end{array} + \frac{r^2 + (z + z_1)^2}{4\alpha t} \right) \right]$$

$$= \frac{Q}{4\pi Kb'} \left[ \frac{2}{\pi} \left( \begin{array}{c} e^{-\beta^2} \frac{r^2 + (z - z_1)^2}{\sqrt{r^2 + (z - z_1)^2}} \right) \frac{1}{\sqrt{r^2 + (z + z_1)^2}} \frac{e^{-\beta_1^2}}{\sqrt{r^2 + (z + z_1)^2}} d\beta_1 d\beta \right]$$

The substitution $\beta = \sqrt{r^2 + (z - z_1)^2} y$ and $\beta_1 = \sqrt{r^2 + (z + z_1)^2} y$

gives $d\beta = \sqrt{r^2 + (z - z_1)^2} dy$, and $d\beta_1 = \sqrt{r^2 + (z + z_1)^2} dy$

Therefore

$$s_{ca} = \frac{Q}{4\pi Kb'} \left[ \frac{2}{\pi} \left( \begin{array}{c} e^{-r^2 y^2} \frac{1}{\sqrt{\alpha t}} \frac{-(z - z_1)^2 y^2}{\sqrt{\alpha t}} \frac{-(z + z_1)^2 y^2}{\sqrt{\alpha t}} \\ 0 \end{array} \right) \right]$$

The substitution $(z - z_1)y = x$ and $(z + z_1)y = x_1$ gives $y dz = dx = dx_1$. 

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\[ s_{ca} = \frac{Q}{4\pi K b'} \left[ \frac{2}{\sqrt{4\alpha t}} \int_{y}^{\infty} e^{-r^2 y^2} dy \left( \int_{0}^{b' - z_1} e^{-x^2} dx + \int_{0}^{b' + z_1} e^{-x_1^2} dx_1 \right) \right] \]

\[ = \frac{Q}{4\pi K b'} \left[ \frac{1}{\sqrt{4\alpha t}} \int_{y}^{\infty} e^{-r^2 y^2} dy \left[ \text{erf} \left( b' - z_1 \right) y + \text{erf} \left( b' + z_1 \right) y \right] \right] \]

The substitution \( r^2 y^2 = \beta \) gives \( 2r^2 y dy = d\beta \). The solution can finally be put in the form

\[ s_{ca} = \frac{Q}{3\pi K b'} \left[ \frac{1}{\beta} \left( \text{erf} \left( \frac{b' - z_1}{r} \sqrt{\beta} \right) + \text{erf} \left( \frac{b' + z_1}{r} \sqrt{\beta} \right) \right) \right] \]

(15)

(b) Steady-state solution

Strictly speaking, a steady-state solution is never attained, as an infinite aquifer is a closed reservoir into which flow from other sources does not take place. However, at any given finite
distance from the center of the spherical cavity, however large it
can be, the drawdown may, for all practical purposes, attain a con-
stant value after a relatively large period of pumping (theoretically
the drawdown is still changing, though at a very slow rate). Thus
as time becomes greater (provided \( R \) remains finite, however large
it may be), the steady state is approached. If \( t \) goes to infinity,
equation (14) reduces to

\[
s_{cm} = \frac{Q}{4\pi K} \left[ \frac{1}{r^2 + (z - z_1)^2} + \frac{1}{r^2 + (z + z_1)^2} \right] \tag{16}
\]

The average steady drawdown in wells of perforated section
can be obtained by integrating equation (16) with respect to \( z \), from
the limits \( 0 \) to \( b' \), and dividing by \( b' \):

\[
s_{cam} = \frac{Q}{4\pi K b'} \left[ \int_{0}^{b'} \left( \frac{1}{r^2 + (z - z_1)^2} + \frac{1}{r^2 + (z + z_1)^2} \right) \, dz \right]
\]

\[
= \frac{Q}{4\pi K b'} \left[ \sinh \left( \frac{b' - z_1}{r} \right) + \sinh \left( \frac{b' + z_1}{r} \right) \right] \tag{17}
\]
Wells Just Tapping the Aquifer

Case 1. Wells of constant discharge

The boundary-value problem of this case is given by equations (1), (2), (3), (4), (11), and the following condition:

$$ s_1 = 0 $$

(a) Nonsteady solution

By applying condition (18) to equation (14), the solution can be found as

$$ s = \frac{Q}{2\pi K} \cdot \frac{1}{R_1} \text{erfc} \left( \frac{R_1}{\sqrt{4\pi \alpha t}} \right) $$

Average drawdown in wells of perforated depth: The average drawdown in a bore hole of length $b'$, can be obtained by applying condition (18) to equation (15). Thus the solution can be put in the form

$$ s_a = \frac{Q}{4\pi K b'} F(u, \frac{b'}{r}) $$

where

$$ F(u, \frac{b'}{r}) = \int_{u}^{\infty} e^{-\beta} \frac{\text{erf} \left( \frac{b'}{r} \sqrt{\beta} \right)}{\beta} d\beta $$
(b) Steady-state solution

As stated before, the steady-state can be approached provided that $R_1$ remains finite, however large it may be. Thus if $t$ goes to infinity, equation (19) reduces to

$$s_m = \frac{Q}{2\pi KR_1}$$  \hspace{1cm} (21)

The average steady drawdown in wells of perforated section $b'$, can be obtained by applying condition (18) to equation (17). The solution can be found as

$$s_{am} = \frac{Q}{2\pi K b'} \sinh^{-1} \left( \frac{b'}{r} \right)$$  \hspace{1cm} (22)

Case 2. Wells of constant drawdown

The boundary-value problem of this case is given by equations (1), (2), (3), (4), and the following two conditions:

$$s_1 = 0$$  \hspace{1cm} (23)

$$s_{(R_w, t)} = s_w$$  \hspace{1cm} (24)
By applying the Laplace transformation on (23) and (24), the transformed conditions in \((R, p)\) will be:

\[
S_1 = 0 \tag{25}
\]

\[
\bar{S} (R_w, p) = \frac{s_w}{p} \tag{26}
\]

Applying condition (25) and then condition (26) to equation (10), one obtains as the value of the constant \(c\):

\[
c = \frac{R_w \sqrt{p}}{\alpha}
\]

\[
c = \frac{\alpha}{R_w} \frac{R_w}{p}
\]

Therefore, the transformed solution is:

\[
\bar{s} = \frac{s_w R_w}{R_1} e^{-\frac{(R_1 - R_w)}{p}} \tag{27}
\]

where \(R_w\) is the effective radius of the spherical cavity at the bottom of the well.
(a) Nonsteady solution

By using tables of the inverse Laplace transform (Churchill 1958), the solution can finally be written as:

\[ s = \frac{s_w}{R_w} \cdot \frac{R_1 - R_w}{\sqrt{\pi \alpha t}} \text{erfc} \left( \frac{R_1 - R_w}{\sqrt{4 \alpha t}} \right) \]  \hspace{1cm} (28)

Discharge of the well

The discharge of the well at any time \( t \) is given by:

\[ Q = -2\pi KR_w^2 \cdot \frac{ds}{dR_1} \quad \text{at} \quad (R_w, t) \]  \hspace{1cm} (29)

By finding \( \frac{ds}{dR_1} \) \((R_w, t)\) from equation (28) and substituting in equation (29), the discharge variation will be given by:

\[ Q = 2\pi K_s s \cdot \frac{R_w}{R_w} \left[ \frac{2}{\sqrt{\pi}} \cdot \frac{R_w}{\sqrt{4 \alpha t}} + 1 \right] \]  \hspace{1cm} (30)

Average drawdown in wells of perforated depth:

By integrating equation (28) with respect to \( s \) between the limits \( a \) and \( b' \), dividing by \( b' \), the average drawdown \( s_a \) in a well of perforated section \( b' \), can be found as:
(b) Steady-state solution

As stated in case 1, the steady state can be approached, provided that \( R_1 \) remains finite, however large it may be. Thus if \( t \) goes to infinity, equation (28) will reduce to:

\[
s_m = \frac{s_w R_w}{R_1}
\]  

(32)

The average steady-state drawdown in an observation well of perforated length \( b' \) can be obtained by integrating equation (32) with respect to \( z \) between the limits \( 0 \) and \( b' \), and dividing by \( b' \):

\[
s_{ma} = \frac{s_w R_w}{b'} \sinh^{-1} \frac{b'}{r}
\]  

(33)

The minimum discharge of the well is that of the steady state. From equation (30), it is:

\[
Q_m = 2\pi K s_w R_w
\]  

(34)
Wells of Finite Depth

The solution for this case is that of spherical cavities each having a discharge \( \frac{Q}{b} \) distributed continuously along the whole depth \( b \) of the well. Thus the boundary-value problem is that given by equations (1), (2), (3), (4), and the following conditions:

\[
\lim_{R \to 0} 4\pi KR^2 \frac{ds}{dR} = -\frac{Q}{b} \quad (35)
\]

\[
s = \int_{0}^{b} s_c \, ds_1 \quad (36)
\]

where \( s_c \) is the drawdown due to a single spherical cavity located at point \( (o, \, z_1) \), as given by equation (10).

If the Laplace transformation is applied to (35) and (36), the transformed condition in \( (R, \, p) \) will be:

\[
\lim_{R \to 0} 4\pi KR^2 \frac{d\bar{s}}{dR} = -\frac{Q}{bp} \quad (37)
\]

\[
\bar{s} = \int_{0}^{b} \bar{s}_c \, ds_1 \quad (38)
\]
By applying condition (37) to equation (10), the constant $c$ can be found as:

$$c = \frac{Q}{4\pi Kbp}$$

With this value of $c$, the transformed solution is:

$$\tilde{s} = \frac{Q}{4\pi Kb} \left[ \frac{1}{p} \cdot e^{-\frac{r}{\alpha} \left\{ \frac{r^2 + (z - z_1)^2}{r^2 + (z - z_1)^2} \right\}} + \frac{1}{p} \cdot e^{-\frac{r}{\alpha} \left\{ \frac{r^2 + (z + z_1)^2}{r^2 + (z + z_1)^2} \right\}} \right]$$

(39)

Applying condition (38) to equation (39), one obtains:

$$\tilde{s} = \frac{Q}{4\pi Kb} \left[ \int_{0}^{b} \frac{1}{p} \cdot e^{-\frac{r}{\alpha} \left\{ \frac{r^2 + (z - z_1)^2}{r^2 + (z - z_1)^2} \right\}} dz_1 + \int_{0}^{b} \frac{1}{p} \cdot e^{-\frac{r}{\alpha} \left\{ \frac{r^2 + (z + z_1)^2}{r^2 + (z + z_1)^2} \right\}} dz_1 \right]$$

(40)
(a) Nonsteady solution

By using tables of the inverse Laplace transform, the draw-down distribution will appear as:

\[
s = \frac{Q}{4\pi Kb} \left[ \int_{0}^{b} \frac{\text{erfc} \left( \frac{\sqrt{r^2 + (z - z_1)^2}}{4\alpha t} \right)}{\sqrt{r^2 + (z - z_1)^2}} \, dz_1 \right]
\]

which, by further simplification, can be expressed as:

\[
s = \frac{Q}{3\pi Kb} \, E(u, \frac{b}{r}, \frac{z}{r})
\]  

(41)

where

\[
E(u, \frac{b}{r}, \frac{z}{r}) = \left[ \int_{u}^{\infty} \frac{e^{-\beta}}{\beta} \cdot \text{erf} \left( \frac{b + z}{r \sqrt{\beta}} \right) \, d\beta \right]
\]

\[
+ \left[ \int_{u}^{\infty} \frac{e^{-\beta}}{\beta} \cdot \text{erf} \left( \frac{b - z}{r \sqrt{\beta}} \right) \, d\beta \right]
\]
Average drawdown in wells of perforated depth:

By integrating equation (41) with respect to $\eta$ between 0 and $b'$, dividing by $b'$, and simplifying, the average drawdown in a well perforated throughout its depth will be found to be:

$$s_a = \frac{Q}{8\pi K b} E\left(\frac{b}{r}, \frac{b'}{r}\right)$$

(42)

where $E(\frac{u}{b'}, \frac{b}{r}) = \left[ 1 + \int_u^\infty \frac{b + b'}{b'} \cdot \frac{e^{-\beta}}{\beta} \cdot \text{erf}(\frac{b + b'}{b'} \sqrt{\beta}) \ d\beta \right] - \frac{2r}{b'} \pi u \left[ \exp \left\{ - \left( \frac{b + b'}{r} \right)^2 u \right\} - \exp \left\{ - \left( \frac{b - b'}{r} \right)^2 u \right\} \right]$

$$- \frac{2r}{b'} \sqrt{\left\{ \frac{b + b'}{r} \right\}^2 + 1} \cdot \text{erfc} \sqrt{\left\{ \left( \frac{b + b'}{r} \right)^2 + 1 \right\} u}$$

$$+ \frac{2r}{b'} \sqrt{\left\{ \frac{b - b'}{r} \right\}^2 + 1} \cdot \text{erfc} \sqrt{\left\{ \left( \frac{b - b'}{r} \right)^2 + 1 \right\} u}$$
(b) Steady-state solution

As stated previously, a steady state can be approached as
time becomes very large, provided that $r$ remains finite. Thus,
as $u$ goes to zero, equation (41) reduces to:

$$s_m = \frac{Q}{4\pi Kb} \left[ \sinh^{-1} \left( \frac{b + b'}{r} \right) + \sinh^{-1} \left( \frac{b - b'}{r} \right) \right]$$  \hspace{1cm} (43)

Similarly, equation (42) will reduce to:

$$s_{am} = \frac{Q}{4\pi Kb} \left[ \frac{b + b'}{b'} \sinh^{-1} \left( \frac{b + b'}{r} \right) - \frac{b - b'}{b'} \sinh^{-1} \left( \frac{b - b'}{r} \right) - \right.$$  

$$\frac{1}{b'} \sqrt{(b + b')^2 + r^2} + \frac{1}{b'} \sqrt{(b - b')^2 + r^2} \right]$$  \hspace{1cm} (44)
DISCUSSION

The nonsteady drawdown distributions around a steady well that penetrates an artesian aquifer, infinite in areal extent, have been obtained by Hantush (1957). For an artesian aquifer of thickness \( m \), the solution is given by either of the following:

\[
s = \frac{Q}{4\pi K m} \left[ W(u) + \frac{2m}{Wb} \sum_{n=1}^{\infty} \frac{1}{n} \cos \frac{nWz}{m} \cdot \sin \frac{nWb}{m} \cdot W_n(u, \frac{nWz}{m}) \right]
\]  

(45)

or

\[
s = \frac{Q}{3\pi K b} \left[ \left( \int_{u}^{\infty} e^{-\frac{\beta}{b}} d\beta \right) \left( \text{erf} \left( \frac{b-s}{r} \sqrt{\beta} \right) + \text{erf} \left( \frac{b+s}{r} \sqrt{\beta} \right) \right) + \right.
\]

\[
\left. \sum_{n=1}^{\infty} \left( \int_{u}^{\infty} e^{-\frac{\beta}{b}} d\beta \right) \left\{ \text{erf} \left( \frac{2nm + b - s}{r} \sqrt{\beta} \right) - \text{erf} \left( \frac{2nm - b + s}{r} \sqrt{\beta} \right) \right\} \right]
\]  

(46)

The flow pattern around a well partially penetrating an artesian aquifer of limited thickness will, during the initial period of flow, closely approximate that of an aquifer of unlimited depth.
The length of the period during which the two flow patterns approach each other depends on the depth of penetration, the distance from the pumped well, and the formation coefficients.

Computations in Hantush's solution and in equation (41) for $z = 0$ show that for relatively short periods of pumping and for values of $\frac{b}{m} < .01$, the two solutions will yield the same results. Thus, for large values of $m$ and/or small values of $x$ (that is, for relatively small values of penetration and/or short distances from the pumped well), equation (41) can be used instead of equation (46), provided that the period of pumping is relatively short. Consequently, it is possible to use equation (41) for determining the formation coefficients by employing data collected during pumping tests on wells that partially penetrate an aquifer of unknown relatively large thickness. Equation (41) can also be used to predict declines of water level during short periods of well operation. For long term prediction, equations (45) or (46) should be used.

Comparing equation (42) with the Theis formula (1935) for an aquifer whose thickness is assumed to coincide with the bottom of the pumped well shows that the two solutions will approach each other during the early stages of pumping only if the distances are
small. They deviate greatly from each other for large distances, although the general trend of the variation appears to be the same. The period during which the two solutions approach each other may range from a few minutes to several months, depending on the distance, thickness of the aquifer, and the formation coefficients. Thus, if it is known that the well partially penetrates a thick aquifer, analysis based on the Theis formula, the aquifer being assumed to end with the bottom of the well, may or may not give reliable results from data collected during a pumping test.

Figure 6 is constructed for a given set of the parameters involved, to illustrate the points discussed above. The values used are \( r = 10 \text{ ft} \), \( b = 100 \text{ ft} \), and \( \frac{r^2 S}{4K} = 1 \), and the ratio of penetration is 0.1.

Figure 7 is constructed to demonstrate the behavior of the drawdown curve given by equation (42) in comparison with the Theis formula for different values of \( r \), the values used are \( r = 1, 10, 100 \text{ ft} \), and \( u = \frac{8 \times 10^{-5}}{t} r^2 \).

In Figure 8, the drawdown observed in a well that is perforated throughout its depth of saturation is compared with the drawdown that would have been observed had the same well been
open at the top of the aquifer alone. It is assumed that \( \frac{b}{r} = 10 \)
\[
\frac{r^2 s}{4k}
\]
and that \( \frac{8}{4k} = 1 \). The same figure also shows the contrasts
in drawdowns for different depths of the observed well. Thus,
in analyzing data observed in a perforated well or in an open hole
(except in the pumped well), equation (42), which gives the av-
erage drawdown, should be used instead of equation (41), which
gives the drawdown at any point \( (r, z, t) \). Apart from the well
losses, the water level in a pumped well that partially penetrates
an aquifer will correspond to that experienced at the top of the
aquifer. Therefore, equation (41), with \( z = 0 \) and \( r = r_w \), is
to be used when predicting water levels in the pumped well or
when analyzing data observed in such a well.
Time drawdown variation with $r = 1, 10, 100$ ft, and $u = \frac{8 \times 10^{-5}}{t} r^2$

- Thickness of the aquifer = m
- Thickness of the aquifer = ∞
Time drawdown variation with $\frac{b}{r} = 10$ and $\frac{r^2}{4K} = 1$. 

Figure 8
TABLES OF FUNCTIONS

To make use of the solutions obtained in the present work, tabular values of the functions $\text{erfc}(x)$, $e^{-x}$, $\sinh^{-1}(x)$, and

$$ F(u, x) = \int_{u}^{\infty} e^{-\beta^2} \text{erf}(x \sqrt{\beta}) \, d\beta $$

should be available. The first three of these functions are available in the literature (Dwight, Mathematical Tables, Dover). For the function $F(u, x)$, previously not available in tabular form, sufficient tabulation for all practical purposes is given in Table 1. It is obtained through numerical integration by using the trapezoidal rule. Although the table is not suitable for linear interpolation, it gives, however, a sufficient number of points to construct smooth curves ($F$ versus $u$, and $F$ versus $x$), that can be used to obtain intermediate values with sufficient accuracy. Figure 9 is a plot for $F(u, x)$ versus $x$.

For $u \leq \frac{0.01}{x^2} > 10$, the function $F(u, x)$ can be approximated by

$$ F(u, x) = (2x - \frac{x^3}{3}) \text{erf} \left( \frac{1}{x} \right) - \text{erf} \left( \sqrt{u} \right) - \frac{2}{\sqrt{\pi}} \left( \frac{x^3}{3} \sqrt{ue^{-u}} - \frac{x}{2} e^{-\frac{u}{x}} \right) $$

(47)
For $\frac{u}{x} \geq \frac{9}{2}$, the approximation is

$$F(u, x) = W(u)$$

(48)

where $W(u)$ is the exponential integral, or what in the field of hydrology is known as the well function. The function has been tabulated in the literature (Wenzel, 1942; Wisler and Brater, 1940).
TABLE 1 - VALUES OF THE FUNCTION $F(u, x)$

\[ F(u, x) = \int_{u}^{\infty} \frac{\eta}{\pi} \text{erf} \left( \frac{\eta x}{F} \right) d\eta \]

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<p>| $u$ | $x$  | 10  | 15  | 20  | 25  | 30  | 35  | 40  | 45  | 50  | 55  | 60  | 65  | 70  | 75  | 80  | 85  | 90  | 95  | 100 |
|-----|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
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| $5 \times 10^{-1}$ | .560 | .219 | .011 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 | .00012 |</p>
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| 10^3  | .560  .219  .0011  .00012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012  .000012 
Figure 9

Type curve of the function $F(u,k)$
APPLICATIONS

In the quantitative study of ground-water resources, it is essential to determine the field values of the so-called formation constants; namely, the hydraulic conductivity and the specific storage. Based on the solutions obtained in each of the flow systems discussed above in the section on theory, methods are outlined below for determining the formation coefficients by using data from an aquifer test. The methods can be classified as:

(1) the type-curve method, (2) the inflection-point method, and
(3) the two-point method.

Type-Curve Method

This method is essentially that of Theis (Wenzel, 1942). The appropriate type curve is a plot of the related function of the flow system versus \( \frac{1}{u} \) on log-log paper. The observational data are plotted against \( t \) on log-log paper of the same scale. The two curves are matched, the axis of the two sheets being kept parallel. A matching point anywhere on the two sheets is selected, and the usual procedure is followed in computing for the formation coefficients.

The systems of flow and their appropriate functions for the type curves are listed below:
<table>
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<tr>
<th>Flow system</th>
<th>Observational well</th>
<th>Function</th>
<th>Coefficient obtained</th>
<th>Equation</th>
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<td>K &amp; $S_s$</td>
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<td>Case 1: Wells of constant discharge</td>
<td>Just tapping the aquifer</td>
<td>$\text{erfc} \left( \sqrt{\frac{u}{r}} \right)$</td>
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<td>K &amp; $S_s$</td>
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<tr>
<td>Case 2: Wells of constant drawdown</td>
<td>Just tapping the aquifer</td>
<td>$\text{erfc} \left( \sqrt{\frac{u}{r-w}} \right)$</td>
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<td>(23)</td>
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<td>Wells of finite depth</td>
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<td>Just tapping the aquifer</td>
<td>$F \left( u, \frac{b}{r} \right)$</td>
<td>K &amp; $S_s$</td>
<td>(41), ($z = 0$)</td>
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<td>$E \left( u, \frac{b}{r}, \frac{b'}{r} \right)$</td>
<td>K &amp; $S_s$</td>
<td>(42)</td>
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</table>

* The coefficients $K$ and $S_s$ can be determined by using equation (30). A plot of $Q$ against $\frac{1}{\sqrt{t}}$ is a straight line whose intercept with the $Q$ axis = $2\pi KR\frac{S_s}{W}$, from which $K$ is obtained. The slope of the straight line

$$= 2\pi KR\frac{S_s}{W} \cdot \frac{R_w}{\sqrt{\pi}} \cdot \sqrt{\frac{S_s}{K}}$$

from which $S_s$ is obtained.
It should be observed that the type curve is not the same for all the observation wells used in an aquifer test; rather, each observation well has its own type curve. This necessitates a lengthy procedure; however, it is the only method yet devised. In actual application, the early part and probably the latter part of the observed data may have to be rejected. During the early period of pumping, the probable variation of the formation coefficients will cause the observed data to deviate from the type curve. The deviation of the data collected during the latter period of pumping may be due to the fact that the flow system is beginning to be influenced by the impermeable bed of the aquifer.

**Inflection-Point Method**

This method can be used only if a steady-state flow is essentially attained and if both the pumped and the observed wells are just tapping the aquifer.

The method is based on the observation that the curve of the time-drawdown semilog plot has an inflection point at $u_1 = 1/2$. This being the case, the drawdown at the inflection point $s_i$ is then equal to $0.317 \frac{s_m}{m}$, where $s_m$ is the maximum drawdown. Therefore,
if the maximum drawdown can be extrapolated from the time-drawdown semilog plot, the point of inflection can be located on the graph by scaling the value \( s_i = 0.317 \, s_m \). The value of \( K \) is obtained by using equation (21), and that of \( S_s \) by using the relation \( u_i = 1/2 \).

If the flow system is that of a flowing well or a spring that just taps the aquifer, the same procedure can be followed, subject to the same conditions, to obtain the effective radius \( R_w \) and the formation coefficients, by using equations (32), (34), and (28) successively.

**Two-Point Method**

This method can be used if the observation well is open only at the top of the aquifer. It can also be used to analyze data from the pumping well, provided that the well losses can be estimated or are very small. The method is based on the following analysis:

Let \( t_1 \) be any convenient time, and let \( n \) be any number (in practice \( n \) is conveniently taken to be 2). Then, if \( s_1 (t_1) \) and \( s_2 (nt_1) \) are the values of the drawdown at time \( t_1 \) and \( nt_1 \), it follows from equation (41) that

\[
\frac{s_1 (t_1)}{s_2 (nt_1)} = \frac{F_1 (u, b/x)}{F_2 (u/n, b/x)} = f(u, \frac{b}{x}).
\]
Here \( \frac{a_1}{s_2} \) and \( \frac{b}{r} \) are known, and so \( u \) can be read from a table or a graph of \( f(u, \frac{b}{r}) \), which function is tabulated in Table 2 for \( n = 2 \). When \( u \) is found, \( F_1(u, \frac{b}{r}) \) is obtained from Table 1.

Hence \( K \) can be computed from

\[
K = \frac{Q}{4\pi b s_1} F_1(u, \frac{b}{r}), \quad \text{and} \quad S_g \text{ from } u = \frac{r^2 s_g}{4Kt_1}.
\]

The procedure is carried out for several pairs of points (usually \( t_1, 2t_1, 3t_1, \ldots \); the computed values of \( K \) and \( S_g \) then are averaged.
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EXAMPLES

Aquifer test data obtained are used to illustrate application of the type-curve method. Two pumped wells have been treated. The effective radius is taken as the radius of the gravel packing, and the well losses are neglected. It should be observed that because of these assumptions, the results of the analysis undoubtedly are in error. The analysis is carried out for the purpose of illustrating the method only.

The first well is located at SE1/4SW1/4SE1/4 sec. 20, T. 11 N., R. 30 E., Quay County, its casing is 12 in. diameter, with 6 in. gravel pack, and has a perforated depth of 30 ft. The well was pumped at an average rate of 12 gallons per minute. Figure 10 is a plot of the type curve using equation (38) with \( z = 0 \). The type curve is matched on the observed drawdown curve. A matching point was chosen having the coordinates; \( F = 1.2; \frac{1}{u} = 25.6; s = 5 \text{ ft}; t = 20 \text{ min.} \) The permeability and specific storage can be found as

\[
K = \frac{114.6 \times 12 \times 1.2}{30 \times 5} = 11.0 \text{ gal/day/ft}^2
\]

\[
S_s = \frac{11 \times 20}{1440 \times 25.6 \times 1.87} = 0.0032 \text{ ft}^{-1}
\]
The second well is located at SE1/4NE1/4SW1/4 sec. 12, T. 24 S., R. 11 W., Luna County, its casing is 12 in. diameter, and has a perforated depth of 100 ft. The well was pumped at an average rate of 374 gal/min. Figure 11 is a plot of the type curve using equation (38) with $a = o$. The type curve is matched on the observed drawdown curve. A matching point was chosen having the coordinates; $F = 4.8; \frac{1}{u} = 2.8 \times 10^{-4}; s = 10$ ft; $t = 100$ min. The permeability and specific storage can be found as

$$K = \frac{114.6 \times 374 \times 4.8}{10 \times 100} = 205.7 \text{ gal/day/ft}^2$$

$$S_s = \frac{205.7 \times 100}{1440 \times 2.8 \times 10^{-4} \times 1.87} = .00028 \text{ ft}^{-1}$$
Fig-10
Type Curve and Time
Drawdown Variation

Type curve

O O O Observed values

Matching point:
F = 1.2 ; 1/u = 25.6
s = 5 ft ; t = 20 min.

Matching point
Fig-11
Type Curve and Time
Drawdown Variation

Type curve

o o o Observed values

Matching point :-
F = 4.8 ; l/u = 2.8 x 10^4
s = 10 ft ; t = 100 min.
REFERENCES


This thesis is accepted on behalf of the graduate faculty of the Institute by the following committee:

Mahdi Mantuah

Christine L. Balk

Martin S. Farrow

William Woo

Clay T. Smith

Maurice B. Volkemer

Date:

May 13, 1960
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*The values of the function for the smaller intervals (not included in Table 1) are obtained by graphical interpolation.*
### TABLE 3  Values of the function \( F(u, x) \) (continued)

| \( x \) | \( u \) | 150  | 200  | 250  | 300  | 350  | 400  | 450  | 500  | 550  | 600  | 650  | 700  | 750  | 800  | 850  | 900  | 950  | 1000 | 1500 | 2000 | 2500 | 3000 | 3500 |
|--------|--------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| \( 1 \times 10^{-8} \) |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 2      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 3      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 4      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 5      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 6      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 7      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 8      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 9      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| \( 1 \times 10^{-7} \) |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 2      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
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| 6      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
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| 9      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| \( 1 \times 10^{-6} \) |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 2      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
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| 9      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| \( 1 \times 10^{-5} \) |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 2      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
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| 5      |       |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |

*The values of the function for the smaller intervals (not included in Table 1) are obtained by graphical interpolation.*