PhD Preliminary Examination in Analysis

Department of Mathematics
New Mexico Tech

Syllabus

The PhD Preliminary Examination in Analysis is intended to determine whether a student has adequate knowledge in the general area of real and complex analysis to begin a research program in applied mathematics. The exam will be written and graded by a committee of professors in the Department of Mathematics with expertise in analysis.

The exam will consist of approximately six to eight questions. The students will be given four hours to take the exam. Notes will not be allowed in the examination. A score of 70% or better will be considered passing. The committee can set a lower passing grade depending on circumstances. Students who fail the exam will be given the chance to take the exam one more time.

Students interested in taking the exam should have taken the courses MATH 435 and MATH 471 or their equivalents. A student should prepare for the exam by studying the relevant material in several of the reference books. Review problems will be provided to the students preparing for the exam. The material covered in the exam will include the following topics:

- **Real Analysis**

  1. Preliminaries.

     Logic, set theory, Cartesian product, relations, functions, countable and uncountable sets, real numbers, supremum, infimum, completeness axiom.
  2. Elements of Point Set Topology.

     Open and closed sets, limit points, compact sets, interior of a set, boundary of a set, closure of a set, Bolzano-Weierstass theorem, Heine-Borel theorem, metric spaces.
  3. Limits and Continuity.

     Convergent sequences, subsequences, Cauchy sequences, complete metric spaces, limit of a function, continuous functions, homeomorphisms, connectedness, uniform continuity, monotonicity.
  4. Differentiation.

     Mean value theorem, intermediate value Theorem, Taylor's theorem, functions of bounded variation, total variation.
  5. Integration.

     Partitions, Riemann and Riemann-Stieltjes integral, upper and lower integrals, mean value theorem, fundamental theorem of calculus, Lebesgue's criterion for Riemann integrability.

6. Sequences and Series of Functions.

   Pointwise convergence, uniform convergence, Cauchy condition for uniform convergence, Weierstrass approximation theorem, $\limsup$, $\liminf$, geometric series, alternating series, absolute convergence, conditional convergence, Weierstrass M-test, the integral test, the ratio test, the root test, Dirichlet test, Abel test, the big O, the little o.

- **Complex Analysis**

  1. Complex Numbers and Elementary Functions.

     Elementary functions, stereographic projection, extended complex plane, limits, continuity, linear fractional transformations, cross ratio.

  2. Analytic Functions and Integration.

     Complex differentiability, analyticity, Cauchy-Riemann equations, multivalued functions, Riemann surfaces, complex integration, Cauchy's theorem, Cauchy integral formula, Liouville's theorem, Morera theorem, maximum modulus theorem.

  3. Sequences, Series and Singularities of Complex Functions.

     Complex sequences and series, Taylor series, Laurent series, singularities of complex functions, analytic continuation, monodromy theorem.

  4. Residue Calculus and Contour Integration.

     Cauchy residue theorem, evaluation of definite integrals by contour integrals, principal value integrals, integrals with branch points, winding numbers, argument principle, Rouché's theorem, fundamental theorem of algebra, open mapping theorem, reflection principle, isolated singularities, meromorphic functions, Casorati-Weierstrass Theorem.

Recommended references

1. W. Rudin, *Principles of Real Analysis*, McGraw-Hill, 1976
2. T. M. Apostol, *Mathematical Analysis*, Addison-Wesley, 1974
3. S. G. Krantz, *Real Analysis and Foundations*, CRC Press, 2004
4. L. Ahlfors: *Complex Analysis*, McGraw-Hill, 1979
5. J. Bak, D. J. Newman, *Complex Analysis*, Springer, 2002
6. M. J. Ablowitz and A. S. Fokas, *Complex Variables : Introduction and Applications*, Cambridge University Press, 2003

7. J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, McGraw-Hill, 2003

Note: Many previous editions of these books available on the market will suffice as well.

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Practice Exam

Real Analysis

1. (a) Let $I = [0,1]$ and $f, g : I \to \mathbb{R}$ be real-valued functions such that f and g are continuous on $[0,1]$ and differentiable on $(0,1)$. Prove that there is a point $c \in (0,1)$ such that

$$f'(c)(g(1) - g(0)) = g'(c)(f(1) - f(0)) \,.$$

   (b) Let $I = [0,1]$ and $f, g : I \to \mathbb{R}$ be real-valued functions such that f and g are continuous on $[0,1]$. Prove that there is a point $c \in (0,1)$ such that

$$f(c) \int_0^1 g(x)dx = g(c) \int_0^1 f(x)dx \,.$$

   (c) Prove or disprove: statement (b) is true if f and g are complex-valued.

2. (a) Let $(a_n)_{n=1}^{\infty}$ be a sequence of non-negative real numbers such that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} < \infty.$$

   Let $(s_n)_{n=1}^{\infty}$ be a sequence defined for all $n \in \mathbb{N}$ by

$$s_n = \sum_{k=1}^{n} a_k \,.$$

   Prove that

$$\lim_{n\to\infty} \frac{s_n}{n^2} = 0.$$

   (b) Use the sequence $a_n = \dfrac{n}{(1 + \log n)}$ to prove that the converse of part (a) does not hold in general.

3. Let $f : [0,\infty) \to \mathbb{R}$ be a bounded, differentiable function satisfying

$$\liminf_{t\to\infty} f(t) = 0 \qquad \text{and } \limsup_{t\to\infty} f(t) = 1.$$

   Prove that there is a sequence $(t_n)_{n=1}^{\infty}$ such that: i) $t_n \overset{n\to\infty}{\longrightarrow} \infty$, ii) $f'(t_n) = 0$ for all $n \in \mathbb{N}$, and iii) $\lim_{n\to\infty} f(t_n) = 1$.

4. Let $(a_n)_{n=1}^{\infty}$ be a bounded nondecreasing sequence of real numbers and $(b_n)_{n=1}^{\infty}$ be a sequence defined by

$$b_n = n(a_{n+1} - a_n).$$

   (a) Prove that $\liminf\limits_{n\to\infty} b_n = 0$.

   (b) Give an example of a sequence (a_n) such that the sequence (b_n) diverges.

5. Let $x > 1$ and $(a_n)_{n=1}^{\infty}$ be a sequence defined by

$$a_1 = x, \qquad a_{n+1} = 2 - \frac{1}{a_n} \qquad \text{for } n \in \mathbb{N}.$$

Show that (a_n) converges, and find the limit.

6. Let $\{a_k\}_{k=0}^{n}$ and $\{b_k\}_{k=0}^{n}$ be two sets of n real numbers, and

$$D = \max_{0\le i\le n} |a_i - b_i|.$$

Let $P(x) = \sum_{k=1}^{n} a_k x^k$ and $Q(x) = \sum_{k=0}^{n} b_k x^k$ be polynomials. Show that if polynomial P has n distinct real roots, then there is an $\varepsilon > 0$ such that if $D < \epsilon$, then the polynomial Q has n distinct real roots.

7. Let $\{a_k\}_{k=1}^{n}$ and $\{b_k\}_{k=1}^{n}$ be two sets of n real numbers. Prove that

$$\left(\sum_{k=1}^{n}(a_k + b_k)^2\right)^{1/2} \le \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}.$$

8. Let $h : \mathbb{R} \to \mathbb{R}$ be a periodic function with period 1, (that is for all $x \in \mathbb{R}$, $h(x+1) = h(x)$), defined for $|x| \le \frac{1}{2}$ by $h(x) = |x|$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined for all $x \in \mathbb{R}$ by

$$f(x) = \sum_{n=1}^{\infty} 4^{-n+1} h(4^{n-1}x).$$

   (a) Show that f is continuous on $\mathbb{R}$.

   (b) Show that f is nowhere monotonic.

9. Let $I = [0,1]$ and $f : I \to \mathbb{R}$ be a function such that: i) $f(0) = f(1) = 0$, ii) f is continuous on $[0,1]$, and iii) f' is bounded in $(0,1)$. Prove that for every $\lambda \in \mathbb{R}$ there is some $c \in (0,1)$ such that $f'(c) = \lambda f(c)$.

10. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$a_n = 2\sqrt{n} - \sum_{k=1}^{n} \frac{1}{\sqrt{k}}.$$

Prove that: i) (a_n) converges, ii) if $p = \lim_{n\to\infty} a_n$, then $p \in (1,2)$.

11. Let $(a_n)_{n\in\mathbb{N}}$ be a real-valued sequence such that

$$a_1 \geq 0 \ , \quad a_2 \geq 0 \ , \qquad \text{and} \qquad a_{n+2} = (a_n a_{n+1})^{1/2} \quad \text{for } n \in \mathbb{N} \, .$$

   (a) Show that (a_n) is convergent.

   (b) Show that $\lim_{n\to\infty} a_n = (a_1 a_2^2)^{1/3}$.

12. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and the series $\sum_{n=1}^{\infty} b_n$ is such that the series $\sum_{n=1}^{\infty}(b_n - b_{n+1})$ converges absolutely. Prove that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

13. Let $A, B \subseteq \mathbb{R}$ be disjoint sets of real numbers, that is, $A \cap B = \emptyset$. Show that: i) If A is compact and B is closed, then there exists $\delta > 0$ such that $|a - b| > \delta$ for any $a \in A$ and for any $b \in B$; ii) If A is closed and B is closed, then the above assertion is false.

14. Let $A \subseteq \mathbb{R}$ be a closed set of real numbers. Prove that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the set of its zero points $F = \{x \in \mathbb{R} \mid f(x) = 0\}$ is precisely A, that is $F = A$.

15. Let $f : [0, \infty) \to \mathbb{R}$ be a function such that: i) f is continuous on $[0, \infty)$, ii) f is differentiable on $(0, \infty)$, iii) $f(0) = 0$, iv) $|f'(x)| \leq |f(x)|$ for all $x > 0$. Prove that $f(x) = 0$ for all $x \in [0, \infty)$.

16. Let I be an open interval and $f : I \to \mathbb{R}$ be a function differentiable on I. Prove that f' is continuous if and only if the inverse image under f' of any point is a closed set.

17. Let $I = [0, 1]$, $E \subseteq I$ be a countable subset of I, and $f : I \to \mathbb{R}$ be bounded on I and continuous on $I \setminus E$. Prove that f is Riemann integrable on I.

18. Let $I = [-1, 1]$ and $f : I \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right), & x \neq 0, \\ 0 & x = 0 \, . \end{cases}$$

   Determine whether f is Riemann integrable on I.

19. Let $I = [0, 1]$ and $f : I \to \mathbb{R}$ be a continuous function on I such that

$$\int_0^1 f(t)p(t) \, dt = 0$$

   for any polynomial p. Prove that $f(x) = 0$ for any $x \in I$.

20. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $\alpha \in \mathbb{R}$, and let A, B and C be the sets defined by $A = \{x \in \mathbb{R} \mid f(x) = \alpha\}$, $B = \{x \in \mathbb{R} \mid f(x) \leq \alpha\}$, and $A = \{x \in \mathbb{R} \mid f(x) < \alpha\}$. Show that the sets A and B are closed and the set C is open.

21. Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f is continuous if and only if the inverse image of any closed set is closed.

22. Prove that the intersection of an arbitrary collection of compact sets in $\mathbb{R}$ is compact.

Complex Analysis

1. Let $\omega : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be a function defined by

$$w(z) = z + \frac{1}{z}.$$

Let S be a set in the complex plane defined by

$$S = \left\{ z : \frac{1}{2} < |z| < 2 \right\}$$

Find and sketch the image $\omega(S)$ of the set S under the map ω.

2. Show that all five roots of the complex polynomial

$$P(z) = z^5 + 6z^3 + 2z + 10$$

lie in the annulus $1 < |z| < 3$.

3. Let C be a circle of radius 3 centered at the origin in the complex plane oriented counterclockwise. Evaluate the integral

$$\oint_C \frac{z^3 + 3z - 1}{z^2 + z - 2} dz.$$

4. Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

5. (a) Show that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

(b) Show that

$$\int_0^\infty \sin(x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

6. Let $a, b \in \mathbb{R}$ be real numbers such that $a > |b|$. Show that

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

7. Let z be a point in the complex plane and C be a circle with the center w and a radius r. Let f be a complex function that is analytic in a simply connected domain that contains the circle C. Show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

8. Let f and g be entire functions with no zeros and having the ratio f/g equal to unity at infinity. Show that they are the same function. That is, if f and g are entire functions such that $f(z) \neq 0$ and $g(z) \neq 0$ for any z and $\lim_{z\to\infty} f/g = 1$, then $f = g$.

9. Let f be a complex function that is analytic and nonzero in a region D in a complex plane. Show that $|f|$ has a minimum value in D that occurs on the boundary of D.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(t)| \leq e^{-a|t|}$ for any $t \in \mathbb{R}$, with some constant $a > 0$. Define a complex function F by

$$F(z) = \int_{-\infty}^{\infty} f(t)e^{izt}dt\,.$$

Find the region in the complex plane where F is analytic.

11. Let C be a circle in a complex plane. Let f be a meromorphic function with N simple zeros and M simple poles inside C. Show

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N - M\,.$$

12. Let $a, t > 0$. Show that

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z}}dz = \frac{1}{\sqrt{\pi t}}\,.$$

13. Let C be a circle in the complex plane oriented counterclockwise, $w \in \mathbb{C}$ be a constant, and f be a complex function such that both f and f' are analytic in a simply connected domain containing the circle C. Let $f(z) \neq w$ for any $z \in C$ and let N be the number of points inside C where $f(z) = w$. Show that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z) - w}dz = N\,.$$