Ph.D. Preliminary Examination in Numerical Analysis
Department of Mathematics
New Mexico Institute of Mining and Technology
February 20, 2021, 8 AM – 12 PM

1. This exam is four hours long.
2. Work out all six problems.
3. Start the solution of each problem on a new page.
4. Number all of your pages.
5. Sign your name on the following line and put the total number of pages.
6. Use this sheet as a coversheet for your papers.

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No. of pages:____________

Problem 1. For solving the initial value problem

$$x' = f(t,x),\ \ x(t_0) = x_0,$$

Heun's method (second-order Runge-Kutta method also known as a modified Euler method) is given by

$$X(t+h) = X(t) + \frac{1}{2}(F_1 + F_2),$$

where

$$\begin{cases} F_1 = hf(t, X(t)), \\ F_2 = h\,f(t+h,\ X + hf(t, X(t))). \end{cases}$$

Derive Heun's method.

Problem 2.

Show that the function

$$f(x) = |x|^{\frac{3}{2}}$$

has a unique fixed point in the interval $|x| \le \frac{1}{3}$.

Problem 3.

Derive the three point Gaussian quadrature rule

$$Af(-\xi) + Bf(0) + Cf(\xi)$$

to approximate the integral

$$\int_{-1}^{1} f(x)\ dx.$$

Problem 4. Let A be a symmetric matrix, let $Q = I - \gamma uu^t$ be a Householder reflector, and let $v = -\gamma Au$. Consider the orthogonal similarity update

$$A_1 = QAQ.$$

a) Find a scalar α such that

$$A_1 = A + vu^t + uv^t + 2\alpha uu^t.$$

b) Let $w = v + \alpha u$. Show that

$$A_1 = A + wu^t + uw^t.$$

c) Using symmetry of A_1, the update $A_1 = A + wu^t + uw^t$ can be implemented by a code

for $j = 2 : n$ **do**
  for $i = j : n$ **do**
    $a_{ij} \leftarrow a_{ij} + w_i u_j + u_i w_j$
  end for

end for

Determine the flop count of this code.

Problem 5.

The 2-norm of a matrix $A \in R^{m\times n}$ is defined by

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2.$$

Let x^* be a unit vector such that

$$\|Ax^*\|_2 = \max_{\|x\|_2=1} \|Ax\|_2.$$

Let $A = U\Sigma V^T$ be an SVD of A with singular values sorted in descending order along the diagonal of Σ, let σ_1 be the largest singular value of A, and let v_1 be the first column of matrix V.

Show that $x^* = v_1$, and that $\|A\|_2 = \sigma_1$. Is x^* necessarily unique?

Hint: Use properties of orthogonal matrices. Prove and use a fact that $\max_{\|z\|_2=1} \|\Sigma z\| = \|\Sigma e_1\| = \sigma_1$, where e_1 is the first column of the identity matrix.

Problem 6. Let $\epsilon < 1$ be a very small positive number. Consider a linear system

$$\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} x = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} \equiv b,$$

and a corresponding linear system with a perturbed right hand side

$$\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} \tilde{x} = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon \end{bmatrix} \equiv \tilde{b}.$$

a) Find the relative error of the perturbed solution $\tilde{x}$ in the ∞-norm. Comment on the magnitude of the error relative to ϵ.

b) Find the relative error of the perturbed right hand side $\tilde{b}$ in the ∞-norm. Comment on the magnitude of the error relative to ϵ.

c) Find the ∞-norm condition number of matrix A, and comment on whether matrix A is well or ill conditioned relative to ϵ.

d) Explain the outcomes in parts a) and b). Is there a problem with solving the original system in a floating point arithmetic for an arbitrary b? If yes, then what causes this problem.

e) Show how to modify the original system to avoid the described problem. Justify your answer by redoing items a), b), and c) for the modified system.